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PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS OF 3-BODY TYPE

Abbas Bahri[1,*]
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Abstract

We study the question of the existence of periodic solutions of Hamiltonian systems of the form:

$$(*) \qquad \ddot{q} + V_q(t, q) = 0$$

where $V = \sum_{1 \le i \ne j}^{3} V_{ij}(t, q_i - q_j)$ with $V(t, \xi)$ T periodic in t and singular in ξ at $\xi = 0$. Under additional technical hypotheses on V (of 3-body type), we prove the functional corresponding to (*) has an unbounded sequence of critical points provided that the singularity of V at 0 is strong enough. These critical points are classical T-periodic solutions of (*). If the singularity at $\xi = 0$ is arbitrary, there still exists at least once generalized T-periodic solutions. Generalized solutions are necessary since collision orbits, i.e. solutions which pass through the singularity, are possible. The proof of these results involves novel topological arguments.

AMS (MOS) Subject Classifications: 34C25, 34C35, 58E05, 58F05, 70F07.

Key Words: 3-body problem, periodic solution, collision.

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Periodic solutions of Hamiltonian systems
of 3-body type

by Abbas Bahri and Paul H. Rabinowitz

§1. Introduction

The study of time periodic solutions of the n-body problem is a classical one. See e.g [1]. Our goal in this paper is to present some new variational approaches of a global nature to a class of problems of 3-body type. To be more precise, consider the system:

$$(1). \qquad m_i \ddot{q}_i + \frac{\partial V}{\partial q_i}(q) = 0, \qquad 1 \leq i \leq 3$$

Here $q_i \in \mathbf{R}^\ell$, $\ell \geq 3$ and $m_i > 0$, $1 \leq i \leq 3$, $q = (q_1, q_2, q_3)$, and $V : F_3(\mathbf{R}^\ell) \to \mathbf{R}$. Here $F_3(\mathbf{R}^\ell)$ is the configuration space

$$F_3(\mathbf{R}^\ell) = \{(q_1, q_2, q_3) \in (\mathbf{R}^\ell)^3 \mid q_i \neq q_j \quad \text{if} \quad i \neq j\}.$$

Since our arguments are valid for any choice of $m_i > 0$, $1 \leq i \leq 3$, for convenience we take $m_i = 1$, $1 \leq i \leq 3$ and write (1) more simply as:

$$(HS) \qquad \ddot{q} + V'(q) = 0.$$

Concerning V, we assume

$$(2) \qquad V = \sum_{\substack{i,j=1 \\ i \neq j}}^{3} V_{ij}(q_i - q_j)$$

where for each i, j, the function V_{ij} satisfies

(V_1) $V_{ij}(x) \in C^2(\mathbf{R}^\ell \backslash \{0\}, \mathbf{R})$,
(V_2) $V_{ij}(x) < 0$,
(V_3) $V_{ij}(q)$ and $V'_{ij}(q) \to 0$ as $|q| \to \infty$,
(V_4) $V_{ij}(q) \to -\infty$ as $q \to 0$,
(V_5) For all $M > 0$, there is an $R > 0$ such that $|q| > R$ implies $V'_{ij}(q) \cdot q > M|V'_{ij}(q)|$,
(V_6) There exists $U_{ij} \in C^1(\mathbf{R}^\ell \backslash \{0\}, \mathbf{R})$ such that $U_{ij}(q) \to \infty$ as $q \to 0$ and $-V_{ij} \geq |U'_{ij}|^2$.

Note that potentials like

(3) $$V(q) = -\sum_{\substack{i,j=1 \\ i \neq j}}^{3} \frac{\alpha_{ij}}{|q_i - q_j|^{\beta_{ij}}}$$

where α_{ij} and β_{ij} are positive and satisfy $(V_1) - (V_5)$. Moreover (V_6) is satisfied if $\beta_{ij} \geq 2$ for all i, j. For the classical 3-body problem, we have $\beta_{ij} = 1, \quad 1 \leq i \neq j \leq 3$.

The significance of hypothesis (V_6) can be seen when (HS) is posed as a variational problem. First we choose $T > 0$ and seek T- periodic solutions of (HS). Let $E = W_T^{1,2}(\mathbf{R}, (\mathbf{R}^\ell)^3)$, the Hilbert space of T-periodic maps from $\mathbf{R}$ into $(\mathbf{R}^\ell)^3$ under the norm:

$$\|q\| = \left(\int_0^T |\dot{q}|^2 dt + [q]^2 \right)^{1/2}$$

where

$$[q] = \frac{1}{T} \int_0^T q(s) ds.$$

The functional corresponding to (HS) is

(4) $$I(q) = \int_0^T \left[\frac{1}{2} |\dot{q}|^2 - V(q) \right] dt.$$

If V satisfies $(V_1) - (V_6)$, then, as will be shown in §2, $q \in \Lambda$ where

$$\Lambda = \{q \in E \mid q_i(t) \neq q_j(t) \text{ for all } i \neq j \text{ and } t \in (0, T]\}.$$

Critical points of I in Λ are then easily seen to be classical solutions of (HS).

Our main result is:

Theorem 1. If V satisfies $(V_1) - (V_6)$, then for each $T > 0$, I possesses an unbounded sequence of critical values.

As will be seen later in the proof of Theorem 1, no explicit use was made of the fact that V is independent of t. Thus we also get the following result:

Theorem 1′. Suppose $V = V(t, q) : \mathbf{R} \times F_3(\mathbf{R}^\ell) \to \mathbf{R}$ is T periodic in t and otherwise satisfies $(V_1) - (V_6)$. Then the functional

(5) $$\int_0^T \left(\frac{1}{2} |\dot{q}|^2 - V(t, q) \right) dt$$

has an unbounded sequence of critical value which are T periodic solutions of

$$\ddot{q} + V_q(t,q) = 0. \tag{6}$$

If (V_6) does not hold, it is possible that $I(q) < \infty$ for $q \in E$ but $q_i(t) = q_j(t)$ for some $i \neq j$ and $t \in [0,T]$. We refer to this possibility as a *collision.* When collisions are possible, critical points of I need not be classical solutions of (HS) and a notion of a generalized solution of (HS) is needed. Following a related situation in [2], we say $q \in E$ is a *generalized T-periodic solution of (HS)* if:

$$\begin{aligned}
&(i) && \mathcal{D} = \{t \in 0,T] \mid q_i(t) = q_j(t) \text{ for some } i \neq j\} \text{ has measure } 0.\\
&(ii) && q \in C^2 \text{ on } [0,T]\backslash\mathcal{D} \text{ and satisfies (HS)},\\
&(ii) && -\int_0^T V(q(t))dt < \infty \text{ and}\\
&(iv) && \tfrac{1}{2}|\dot{q}(t)|^2 + V(q(t)) \equiv \text{ constant for } t \in [0,T]\backslash\mathcal{D}\\
& && \text{(i.e. energy is conserved on the set on which it is defined).}
\end{aligned} \tag{7}$$

Theorem 1, together with some of the ingredients in its proof and ideas from [2] yields

Theorem 2. If V satisfies $(V_1) - (V_5)$, for each $T > 0$, (HS) possesses a generalized T-periodic solution.

There is also an analogue of Theorem 2 for the case in which $V = V(t,q)$ and is T-periodic in t.

Our approach to (HS) is via the calculus of variations. A few recent papers [3-6] have used variational methods to treat singular Hamiltonian systems but for potential energy terms which have a milder singularity than (2). E.g. [3-6] study (HS) for V's having a point singularity like $V(q) = W(|q|)$ where $W(s) = -s^{-\beta}$. More generally they treat V's having a compact set of singularities. They also have restrictions on the behavior of V near the singular set like (V_6). Under such hypotheses, the functional corresponding to I satisfies some version of the Palais-Smale condition - (PS) for short - and this fact plays an important role in the associated existence arguments. In work in progress, Coti-Zelati is studying a class of time independent potentials of n body type under a symmetry condition $(V_{ij}(\xi) = V_{ji}(\xi))$. This symmetry and a clever observation allow him to work in a restricted class of functions where (PS) holds. However, in the current setting, under $(V_1) - (V_6)$ the functional defined by (2) and (4) does not satisfy (PS) even after

eliminating a translational symmetry inherent in the form of (2). Roughly speaking, what goes wrong with (PS) is that a sequence $(q^j) \subset E$ with $I(q^j) \to c$ and $I'(q^j) \to 0$ may "approach" the triple $(q_1, q_2, \text{"}\infty\text{"})$ which is a solution of the two body problem associated with (HS) by dropping all terms involving q_3.

To briefly outline the remainder of this paper, the breakdown of (PS) will be studied in a precise way in §2. Invariance properties of I and the behavior of level sets of I, in particular of $I^\epsilon \equiv \{q \in E \mid I(q) \leq \epsilon\}$ for small ϵ will also be examined. A novel kind of Morse Lemma for neighborhoods of infinity will be given in §3. This lemma combined with the results of §2 gives (modulo translations) a priori bounds for critical points of I, the bounds depending on the corresponding critical values. In §4, it will be shown that I can be approximated by a nearly functional $\tilde{I}$ with nondegenerate critical points (modulo translations) and possessing other nice properties.

The proof of Theorem 1 will be carried out in §5 by means of an indirect argument in which I is replaced by $\tilde{I}$. A key role in the proof is played by a notion of critical points at infinity, corresponding to limit two body problems arising from the breakdown of (PS), together with their unstable manifolds. As will be shown in §7-8, $\tilde{I}^\mu$ can be retracted by deformation to $\tilde{I}^{\epsilon_1} \cup \mathcal{D}_M \cup \mathcal{D}_M^\infty$ where $\mathcal{D}_M$ is the union of all unstable manifolds of critical points of $\tilde{I}$ in $\tilde{I}^M \backslash \tilde{I}^{\epsilon_1}$ and $\mathcal{D}_M^\infty$ a similar set for critical points of the limit 2-body problems at infinity. This enables us to exploit the difference in topology as measured by rational homology between Λ and its two body analogue. In §6 we prove Theorem 2. Lastly in §9 under certain assumptions of nondegeneracy of critical points (up to translations), we obtain Morse type inequalities for critical points (Theorem 3). One consequence of these inequalities, which will be pursued elsewhere, is that they enable us to conclude that in certain situations, e.g. for simple potentials where one has central configurations (satisfying (V_6)) that the family of periodic solutions we find is much larger than the known family of solutions. Moreover, using these inequalities and ideas which can be found in Klingenberg [18] and Ekeland [19], under generic conditions one can establish the existence of either an elliptic orbit or infinitely many hyperbolic orbits on a given energy surface (see A. Bahri, B. M. D'Onofrio [20]). If we drop (V_6), these inequalities do not hold per se. However under additional assumptions on V, one can show there are at most finitely many collisions. This fact can be used to prove an analogue of Theorem 3 when collisions can occur and likewise leads to applications such as those just mentioned. These matters will also be pursued elsewhere.

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§2. Some analytic preliminaries

In this section, several of the properties of I will be studied, especially the breakdown of the Palais-Smale condition. For simplicity here and in the sequel we assume the period $T = 1$. To begin we will show that if $q \in E$ and $I(q) < \infty$, then $q \in \Lambda$. More precisely, we have:

Proposition 2.1. Suppose V satisfies $(V_1), (V_2), (V_4)$ and (V_6). Then for any $c > 0$, there exists $\delta = \delta(c)$ such that $q \in E$ and $I(q) \leq c$ implies

$$\inf_{i \neq j, t \in [0,1]} |q_i(t) - q_j(t)| \geq \delta.$$

Proof. Consider two distinct indices i $j \in \{1, 2, 3\}$. By (V_2).

$$-\int_0^1 V_{ij}(q_i(t) - q_j(t))dt \leq c.$$

Since $I(q) < \infty$, by (V_4) there exists $\delta_1 = \delta_1(c) > 0$ and $\tau \in [0, 1]$ such that $|q_i(\tau) - q_j(\tau)| \geq \delta_1(c)$. It may be assumed that $|q_i(\tau) - q_j(\tau)| = \delta_1$ for otherwise the loop $q_i - q_j$ remains outside a neighborhood of 0 radius $\delta_1(c)$ and Proposition 2.1 is proved with $\delta(c) = \delta_1(c)$. Observe that by (V_2) again, $\|\dot{q}_i - \dot{q}_j\|_{L^2} \leq \sqrt{2c}$. Using (V_2) and (V_6), for any $\sigma \in [0, 1]$,

$$\begin{aligned}
c &\geq \left| \int_\tau^\sigma V_{ij}(q_i - q_j)dt \right| \\
&\geq \left| \int_\tau^\sigma |\nabla U_{ij}(q_i - q_j)|^2 dt \right| \\
&\geq \frac{1}{2c} \left| \int_\tau^\sigma |\nabla U_{ij}(q_i - q_j)|^2 dt \right| \int_\tau^\sigma |\dot{q}_i - \dot{q}_j|^2 dt \\
&\geq \frac{1}{2c} \left(\int_\tau^\sigma \nabla U_{ij}(q_i - q_j) \cdot (\dot{q}_i - \dot{q}_j) dt \right)^2 \\
&= \frac{1}{2c} |U_{ij}(q_i(\sigma) - q_j(\sigma)) - U_{ij}(q_i(\tau) - q_j(\tau))|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
U_{ij}(q_i(\sigma) - q_j(\sigma)) &\leq \sqrt{2}c + U_{ij}(q_i(\tau) - q_j(\tau)) \\
&\leq \sqrt{2}c + \sup_{|x| = \delta_1(c)} U_{ij}(x) < \infty.
\end{aligned}$$

This last inequality together with (V_4) and (V_6) implies the result.

Proposition 2.1 allows us to seek critical points of I in Λ and thereby exploit the topological structure of Λ. This will be done in §5. The breakdown of (PS) will be studied in the next proposition. This will lead us to define "critical points at infinity" and their "unstable manifolds" in later sections.

Proposition 2.2. Let V satisfy (V_1) -- (V_4) and (V_6). Let $(q^k) \subset \Lambda$ be a sequence such that $I(q^k) \to c$ and $I'(q^k) \to 0$. Then either

1° there exists a subsequence, again denoted by (q^k) and a sequence $(v_k) \subset \mathbf{R}^\ell$ such that $q_i^k - v_k$ converges in $W^{1,2}$ for $i = 1, 2, 3$, or

2° there exists a subsequence, again denoted by (q^k), $i \in \{1,2,3\}$, and $(v_k) \subset \mathbf{R}^\ell$ satisfying

a. $|[q_i^k] - v_k| \to \infty$, $\|\dot{q}_i^k\|_{L^2} \to 0$, and

b. if $j, r \in \{1,2,3\} \backslash \{i\}$, $(q_j^k - v_k, q_r^k - v_k)$ converges in $W^{1,2}$ to a classical solution of the two-body problem with potential $V_{jr} + V_{rj}$ and as $k \to \infty$,

$$\int_0^1 \left[\frac{1}{2}(|\dot{q}_j^k|^2 + |\dot{q}_r^k|^2) - V_{jr}(q_j^k - q_r^k) - V_{rj}(q_r^k - q_j^k) \right] dt \to c$$

Remark 2.3. In fact we will show $\frac{1}{2}[q_j^k + q_r^k]$ is a permissable choice for v_k.

Proof of Proposition 2.2. As in Proposition 2.1, the bounds on $I(q^k)$ lead to bounds depending on c for $\|\dot{q}^k\|_{L^2}$ and

$$-\int_0^1 V_{ij}(q_i^k - q_j^k)dt, \qquad 1 \le i \ne j \le 3.$$

By Proposition 2.1, there is a $\delta(2c) > 0$ such that

$$|q_i^k(\tau) - q_j^k(\tau)| \ge \delta(2c) \tag{2.4}$$

for all $k \in \mathbf{N}$, $\tau \in [0,1]$ and $i \ne j$. The bounds on $\|\dot{q}^k\|_{L^2}$ and standard embedding theorems imply that $q^k - [q^k]$ converge along a subsequence - which will still be denoted by q^k - weakly in E and strongly in L^∞ to $q \in \Lambda$. If for some r, j, $|[q_j^k - q_i^k]| \to \infty$,

$$V'_{jr}(q_j^k - q_r^k) \to 0 \tag{2.5}$$

in L^∞ via (V_3). If $[q_j^k - q_r^k]$ is bounded, $V'_{jr}(q_j^k - q_r^k)$ converges via (V_1) and (2.4). Thus $V'_{jr}(q_j^k - q_r^k)$ converges for all pairs $r \ne j$ and

$$I'(q^k) \to 0 \tag{2.6}$$

then implies $\dot{q}^k$ converges in L^2 to $\dot{q}$.

If $|[q_j^k - q_r^k]| \to \infty$ for all 3 pairs of indices $j \ne r$, (2.5)-(2.6) show $\dot{q} = 0$ and $I(q^k) \to 0$, a contradiction. Thus there is at least one pair of indices $j \ne r$ such that $[q_j^k - q_r^k]$ is bounded. Without loss of generality we can assume $[q_j^k - q_r^k]$ converges. Set

$$v_k = \frac{1}{2}[q_j^k + q_r^k]. \tag{2.7}$$

Then

$$q_r^k - v_k = q_r^k - [q_r^k] + \frac{1}{2}[q_r^k - q_j^k]$$

converges in $W^{1,2}$ as does $q_j^k - v_k$. Let $i \in \{1,2,3\}\backslash\{j,r\}$. Either (i) $([q_r^k - q_i^k])$ is bounded, or (ii) $\|[q_r^k - q_i^k]\| \to \infty$ (along a subsequence). If (i) holds, we may assume $[q_r^k - q_i^k]$ converges and therefore $q_i^k - v_k$ converges in $W^{1,2}$. This corresponds to case 1° of Proposition 2.2. If (ii) occurs, by (2.6), $V'_{ir}(q_i^k - q_r^k)$, $V'_{ri}(q_j^k - q_r^k)$, $V'_{ji}(q_r^k - q_i^k)$, and $V'_{ij}(q_i^k - q_j^k) \to 0$ as $k \to \infty$ since their arguments $\to \infty$ uniformly in τ as $k \to \infty$. Thus $q_i^k - [q_i^k]$ converges to 0 in $W^{1,2}$, i.e. $\|\dot{q}_i^k\|_{L^2} \to 0$ as $k \to \infty$, and $\|[q_i^k - v_k]\| \to \infty$ as $k \to \infty$. This is precisely case 2° of Proposition 2.2. The proof is complete.

Corollary 2.8. Suppose $q \in \Lambda$ satisfies $0 < a \le I(q) \le b$. Then there exists ϵ_0, $c_0 > 0$ (independent of q), $v(q) \in \mathbf{R}^\ell$, and two indices $j \ne r \in \{1,2,3\}$ such that if $\|I'(q)\|_{W^{1,2}} \le \epsilon_0$, then

$$\|q_j - v\|_{W^{1,2}} + \|q_r - v\|_{W^{1,2}} \le c_0.$$

Proof. This follows immediately from Proposition 2.2.

For $s > 0$, let

$$I^s = \{q = (q_1, q_2, q_3) \in \Lambda | I(q) \le s\}.$$

We now study I^s for small s.

Proposition 2.9. Let V satisfy $(V_1) - (V_5)$. Then there exists an $\epsilon_1 > 0$ such that

(i) $I'(q) \ne 0$ for all $q \in I^{2\epsilon_1}$.

(ii) For all $\lambda \ge 1$ and $q \in I^{\epsilon_1}$,

$$q_\lambda \equiv \lambda[q] + \lambda^{-1}(q - [q]) \in I^{\epsilon_1},$$

(iii) For all $\lambda \in [1,2)$ and $q \in I^{\epsilon_1}$,

$$\overline{q}_\lambda \equiv \lambda[q] + \exp(1 + (\lambda - 2)^{-1}) \quad (q - [q]) \in I^{\epsilon_1}$$
$$\overline{q}_2 \equiv 2[q] \in I^{\epsilon_1},$$

(iv) there is an $\epsilon_2 < \epsilon_1$ such that if $0 < \epsilon \le \epsilon_2$, I^{ϵ_1} is homotopy equivalent to the set

$$\mathcal{B}(\epsilon) \equiv \{(x_1, x_2, x_3) \in (\mathbf{R}^\ell)^3 | -V(x_1, x_2, x_3) \le \epsilon\}.$$

Proof. Without loss of generality, $\epsilon_1 < 1$. Since $(q_\lambda)_\mu = q_{\lambda\mu}$ for all $\lambda, \mu \ge 1$, if $q \in I^{\epsilon_1}$ and

$$\frac{d}{d\lambda} I(q_\lambda)\Big|_{\lambda=1} < 0, \tag{2.10}$$

it easily follows that $q_\lambda \in I^{\epsilon_1}$ for all $\lambda \geq 1$. Hence (ii) and also (i) follow from (2.10). Now

$$\frac{d}{d\lambda} I(q_\lambda)\Big|_{\lambda=1} = -\int_0^1 |\dot{q}|^2 dt - \sum_{i \neq j=1}^{3} \int_0^1 V'_{ij}(q_i - q_j) \cdot \\ \cdot ([q_i - q_j] - (q_i - [q_i] - q_j + [q_j]))dt. \tag{2.11}$$

We rewrite each term in the V sum as

$$-\int_0^1 V'_{ij}(q_i - q_j) \cdot (q_i - q_j)dt + 2\int_0^1 V'_{ij}(q_i - q_j) \cdot \\ \cdot (q_i - [q_i] + [q_j] - q_j)dt. \tag{2.12}$$

Since $I(q) \leq \epsilon_1$, by (V_2) we have

$$\|q_i - [q_i]\|_{L^\infty} + \|q_j - [q_j]\|_{L^\infty} \leq \|\dot{q}_i\|_{L^2} + \|\dot{q}_j\|_{L^2} \leq 2\sqrt{\epsilon_1}. \tag{2.13}$$

Now (2.12)-(2.13) imply

$$\frac{d}{d\lambda}\left(-\int_0^1 V_{ij}((q_\lambda)_i - (q_\lambda)_j)dt\right)\Big|_{\lambda=1} \\ \leq \int_0^1 [-V'_{ij}(q_i - q_j) \cdot (q_i - q_j) + 4\sqrt{\epsilon_1}|V'_{ij}(q_i - q_j)|]dt. \tag{2.14}$$

Applying (V_5), there exists a constant A such that

$$-V'_{ij}(x) \cdot x + 4|V'_{ij}(x)| < 0 \tag{2.15}$$

for $|x| \geq A$. Since $I(q) \leq \epsilon_1$,

$$-\int_0^1 V_{ij}(q_i - q_j)dt \leq \epsilon_1.$$

For

$$\epsilon_1 < \alpha_0 \equiv \min_{|x| \leq 2A}(-V_{ij}(x)),$$

by (V_1) and (V_3), there is a $\tau_1 \in [0,1]$ such that

$$|q_i(\tau_1) - q_j(\tau_1)| \geq 2A. \tag{2.16}$$

Now (2.16) and (2.13) imply for $\epsilon_1 < \frac{A^2}{4} \equiv \alpha_1$ that

$$|q_i(\tau) - q_j(\tau)| \geq A \tag{2.17}$$

for all $\tau \in [0,1]$. Therefore if $\epsilon_1 < \min(\alpha_0, \alpha_1)$, we have

$$-V'_{ij}(q_i(\tau) - q_j(\tau)) \cdot (q_i(\tau) - q_j(\tau)) + 4\sqrt{\epsilon_1}|V'_{ij}(q_i(\tau) - q_j(\tau))| < 0 \tag{2.18}$$

for all $\tau \in [0,1]$. Thus

$$\frac{d}{d\lambda}\left(-\int_0^1 V_{ij}((q_\lambda)_i - (q_\lambda)_j)dt\right)\Bigg|_{\lambda=1} < 0 \tag{2.19}$$

and (2.10), (ii), and (i) follow.

To prove (iii), we need to calculate the derivative of $I(\overline{q}_\lambda)$ for each λ. Using the fact that $\exp(\lambda-2)^{-1}$ and $(\lambda-2)^{-2}\exp(\lambda-2)^{-1}$ are bounded for $\lambda \in [1,2)$, uniformly in λ, the proof is essentially the same as for (ii) and will be omitted.

Lastly we turn to the proof of (ii). Let

$$B = \{2[q] \mid q \in I^{\epsilon_1}\}. \tag{2.20}$$

Using (iii), we can define a homotopy between I^{ϵ_1} and B via

$$\begin{aligned} [0,1] \times I^{\epsilon_1} &\to I^{\epsilon_1} \\ (\theta, q) &\to \overline{q}_{1+\theta}. \end{aligned} \tag{2.21}$$

The continuity of this retraction is clear. Observe now that by (V_1) and (V_3), for ϵ small enough, e.g. $\epsilon \le \epsilon_1$, the set

$$\mathcal{B}(\epsilon) = \{(x_1, x_2, x_3) \in (\mathbf{R}^\ell)^3 \mid -V(x_1, x_2, x_3) \le \epsilon\}$$

is contained in B. Also by (ii), $\lambda b \in B$ if $b \in B$ and $\lambda \ge 1$; indeed if $b = 2[q]$, then $\lambda b = 2[q_\lambda]$. Now (V_1), (V_2), and (V_4), together with the fact that $-V(b) \le \epsilon_1$ for all $b \in B$ (which is a consequence of (iii)) imply that $\phi_b(\lambda) \equiv -V(\lambda b)$ decreases monotonically to 0 on B as $\lambda \to \infty$. Therefore it is possible to define a function $\tilde{\lambda} : B \to \mathbf{R}$ via

$$\tilde{\lambda}(b) = \inf\{\lambda \ge 1 \mid -V(\lambda b) \le \epsilon\}. \tag{2.22}$$

Since $-V(\lambda b)$ decreases monotonically to 0 as $\lambda \to \infty$, $\tilde{\lambda}$ is continuous. The map $u : [0,1] \times B \to B$ defined by

$$u(\theta, b) = (1-\theta)b + \theta\tilde{\lambda}(b)b \tag{2.23}$$

retracts B by deformation on $\mathcal{B}(\epsilon)$ and (iv) holds.

To prepare for the next result, note that I possesses an $\mathbf{R}^\ell$ symmetry. More precisely, for $\xi \in \mathbf{R}^\ell$, let $\psi(\xi) = (\xi, \xi, \xi)$. Then

$$I(q + \psi(\xi)) = I(q) \tag{2.24}$$

for all $q \in \Lambda$ and $\xi \in \mathbf{R}^\ell$. Therefore

$$I'(q)\psi(\epsilon) = 0 \tag{2.25}$$

for all $q \in \Lambda$ and $\xi \in \mathbf{R}^\ell$. Letting D denote the duality map from E' to E, (2.25) is equivalent to

$$\sum_{i=1}^{3} [DI'(q)]_i = 0. \tag{2.26}$$

Now we have:

Proposition 2.27. Let $\Phi = (\Phi_1, \Phi_2, \Phi_3) \in C^1(\Lambda, \Lambda)$ such that

$$\sum_{i=i}^{3} [\Phi_i(q)] = 0 \tag{2.28}$$

for all $q \in \Lambda$. Let $\eta(s, q) = (\eta_1(s, q), \eta_2(s, q), \eta_3(s, q))$ denote the solution of the differential equation

$$\frac{d\eta}{ds} = \Phi(\eta), \qquad \eta(0, q) = q \in \Lambda. \tag{2.29}$$

Then for all s for which the solution is defined,

$$\sum_{i=1}^{3} [\eta_i(s, q)] = \sum_{i=1}^{3} [q_i]. \tag{2.30}$$

Proof. By (2.28)-(2.29),

$$\sum_{i=1}^{3} \left[\frac{d\eta_i}{ds}\right] = 0 = \frac{d}{ds} \sum_{i=1}^{3} [\eta_i]. \tag{2.31}$$

Therefore

$$\sum_{i=1}^{3} [\eta_i(s, q)] \tag{2.32}$$

is independent of s. Hence (2.30) follows from (2.29).

Now some properties of the "two body problem" with potential $V_{ij} + V_{ji}$ will be considered. For the sake of simplicity we take $i = 1$ and $j = 2$. Define

$$\Lambda_{12} \equiv \{(q_1, q_2) \in W_1^{1,2}(\mathbf{R}, (\mathbf{R}^\ell)^2) \mid (q_1(t) \neq q_2(t) \text{ for all } t \in [0,1]\} \tag{2.33}$$

and

$$\begin{aligned} I_{12}(q_1, q_2) &\equiv I_{12}(q) \\ &= \int_0^1 \left[\frac{1}{2}(|\dot{q}_1|^2 + |\dot{q}_2|^2) - (V_{12}(q_1 - q_2) + V_{21}(q_2 - q_1))\right] dt. \end{aligned} \tag{2.34}$$

Propositions 2.1, 2.2, and 2.9 have the following analogues for the two body problem associated with (2.34):

Proposition 2.1′. Let V_{12} satisfy $(V_1), (V_2), (V_4)$ and (V_6). Then for any $c > 0$, there exists $\delta = \delta(c) > 0$ such that $q \in \Lambda_{12}$ and $I_{12}(q) \leq c$ implies

$$\inf_{t \in [0,1]} |q_1(t) - q_2(t)| \geq \delta.$$

Proposition 2.2′. Let V_{12} satisfy $(V_1) - (V_4)$ and (V_6). Let $(q^k) \subset \Lambda_{12}$ be a sequence such that $I_{12}(q^k) \to c > 0$ and $I'_{12}(q^k) \to 0$. Then there exists a subsequence, again denoted by (q^k) and $(v_k) \subset \mathbf{R}^\ell$ such that $q_i^k - v_k$ converges in $W^{1,2}$ for $i = 1, 2$.

Proposition 2.9′. Let V_{12} satisfy $(V_1) - (V_5)$. Then for ϵ_1 small enough, (i) -(iv) of Proposition 2.9 hold with I replaced by I_{12} and $\mathcal{B}(\epsilon)$ by

$$\mathcal{B}_{12}(\epsilon) \equiv \{(x_1, x_2) \in (\mathbf{R}^\ell)^2 \mid -\tilde{V}_{12}(x_1, x_2) \leq \epsilon\}$$

where $\tilde{V}_{12} = V_{12} + V_{21}$.

The proofs of these results follow the same lines as their earlier analogues and will be omitted. Note that Proposition 2.2′ says that I_{12} satisfies the Palais-Smale condition up to translations. Ca[illegible] of Proposition 2.2 has no analogue here since if e.g. $|[q_1^k]| \to \infty$ while $|[q_2^k]|$ remains bounded, then $(q_1^k - [q_1^k])$ and $(q_2^k - [q_2^k])$ converge to zero and therefore $I_{12}(q^k) \to 0$, contrary to hypothesis.

We also have an analogue of Proposition 2.27 with the same proof:

Proposition 2.27′. Let $\Phi_{12} = (\Phi_1, \Phi_2) \in C^1(\Lambda_{12}, \Lambda_{12})$ such that

$$\sum_{i=1}^{2} [\Phi_i(q)] = 0 \tag{2.28′}$$

for all $q \in \Lambda_{12}$. Let $\eta_{12}(s,q)$ denote the solution of the differential equation

$$\frac{d\eta_{12}}{ds} = \Phi_{12}(\eta_{12}), \qquad \eta_{12}(0,q) = q \in \Lambda_{12}. \tag{2.29'}$$

Then for all s for which the solution is defined,

$$\sum_{i=1}^{2}[\eta_{12}(s,q)]_i = \sum_{i=1}^{2}[q_i]. \tag{2.30'}$$

Our final result in this section concerns the following important special case of (2.29'):

$$\frac{d\eta_{12}}{ds} = -I'_{12}(\eta_{12}), \qquad \eta_{12}(0,q) = q. \tag{2.35}$$

For $0 < a \le b < \infty$, set

$$\mathcal{K}_{12}(a,b) \equiv \{q \in \Lambda_{12} \mid I'_{12}(q) = 0 \text{ and } a \le I_{12}(q) \le b\}$$

Proposition 2.36. Let q satisfy $a \le I_{12}(q) \le b < \infty$ and let $\eta_{12}(s,q)$ be a solution of (2.35). Then

(i) there exists a constant $c(q)$ independent of s such that $|[\eta_{12}(s,q)]| \le c(q)$ for any s for which

$$a \le I_{12}(\eta(s,q)) \le b. \tag{2.37}$$

(ii) There exists a constant $\tilde{C}(a,b)$ and a uniform ρ-neighborhood, $N(\rho)$, of $\mathcal{K}_{12}(a,b)$ such that whenever $q \in N(\rho)$, there is a $v(q) \in \mathbf{R}^\ell$ satisfying $\|(\eta_{12}(s,q)) - v(q)\|_{W^{1,2}} \le \tilde{C}(a,b)$, $i = 1,2$, for all $s \ge 0$ for which (2.37) holds.

Proof. Arguing indirectly, assume there exists a sequence s_n for which (2.37) holds and $|[\eta_{12}(s_k,q)]| \to \infty$. Then

$$\|\eta_{12}(s_k,q) - q\|_{W^{1,2}} \le \left| \int_0^{s_k} \|I'_{12}(\eta_{12}(s,q))\|_{W^{1,2}} ds \right| \to \infty \tag{2.38}$$

while

$$\left| \int_0^{s_k} \|I'_{12}(\eta_{12}(s,q))\|^2_{W^{1,2}} ds \right| = \left| \int_0^{s_k} \frac{d}{ds} I(\eta_{12}(s,q)) ds \right| \le b - a. \tag{2.39}$$

If (s_k) were bounded, (2.38) - (2.39) would be contradictory. Therefore

$$\lim_{k\to\infty} s_k = \pm\infty$$

and (2.38) implies the existence of a sequence $\tau_k \to \infty$ satisfying (2.37) and such that $I'_{12}(\eta_{12}(\tau_k, q)) \to 0$. Using Proposition 2.2', there exists $(v_k) \subset \mathbf{R}^\ell$ for which $(\eta_{12}(\tau_k, q))_i - v_k$ is strongly convergent, $i = 1, 2$. Proposition 2.27' then implies that (v_k) is convergent. Therefore $(\eta_{12}(\tau_k, q))$ is convergent.

Assume, without loss of generality, that

$$\lim_{k \to \infty} s_k = \infty.$$

The argument just given shows the existence of γ and $M > 0$ such that if $s \in [0, \infty)$ and

$$\|I'_{12}(\eta_{12}(s, q))\|_{W^{1,2}} \leq \gamma, \tag{2.40}$$

then $\|\eta_{12}(s, q)\|_{W^{1,2}} \leq M$. Since $\|[\eta_{12}(s_k, q)]\| \to \infty$ as $k \to \infty$, (2.40) is violated when $s = s_k$ for large k. Given s_k, let $\overline{s}_k$ be the largest positive value of s less than s_k such that (2.40) holds. The existence of $(\overline{s}_k)$ follows from that of (τ_k). Observe that

$$\|I'_{12}(\eta_{12}(s, q))\|_{W^{1,2}} \geq \gamma \tag{2.41}$$

for $s \in [\overline{s}_k, s_k]$. Now by (2.41) and (2.39),

$$\begin{aligned} \|[\eta_{12}(s_k, q) - \eta_{12}(\overline{s}_k, q)]\| &\leq \int_{\overline{s}_k}^{s_k} \|I'_{12}(\eta_{12}(s, q))\|_{W^{1,2}} ds \\ &\leq \frac{1}{\gamma} \int_{\overline{s}_k}^{s_k} \|I'_{12}(\eta_{12}(s, q))\|^2_{W^{1,2}} ds \leq \frac{b - a}{\gamma}. \end{aligned} \tag{2.42}$$

But $\|[\eta_{12}(\overline{s}_k, q)]\| \leq M$ since (2.40) holds for $s = \overline{s}_k$. Therefore

$$\|[\eta_{12}(s_k, q)]\| \leq M + \frac{b - a}{\gamma} \tag{2.43}$$

contrary to the choice of s_k and (i) follows.

To prove (ii), an argument as in the proof of Proposition 2.2' shows there is a $\rho > 0$ and $\epsilon_1(a, b)$ such that for any $q \in N(\rho)$ there exists $v(q) \in \mathbf{R}^\ell$ satisfying

$$\|q_i - v(q)\|_{W^{1,2}} \leq C_1(a, b), \qquad i = 1, 2. \tag{2.44}$$

The constant $C_1(a, b)$ is independent of $q \in N(\rho)$. Equation (2.35) shows

$$\eta_{12}(s, q - \psi_{12}(v(q))_i = \eta_{12}(s, q)_i - v(q), \qquad i = 1, 2. \tag{2.45}$$

Now arguing indirectly, we assume there exists a sequence $(q^k) \subset N(\rho)$ and $s_k \geq 0$ such that

$$(2.46) \qquad b \geq I_{12}(\eta_{12}(s_k, q^k)) \geq a \quad \text{and} \quad \|\eta_{12}(s_k, q^k) - \psi_{12}(v(q^k))\|_{W^{1,2}} \to \infty \quad \text{as} \quad k \to \infty.$$

Since $I'_{12}(q) = I'_{12}(q - \psi_{12}(\xi))$ for all $\xi \in \mathbf{R}^\ell$, as in (2.38)-(2.39) we have:

$$(2.47) \qquad \int_0^{s_k} \|I'_{12}(\eta_{12}(s, q^k))\|^2_{W^{1,2}} ds \leq b - a;$$
$$\int_0^{s_k} \|I'_{12}(\eta_{12}(s, q^k))\|_{W^{1,2}} ds \to \infty \quad \text{as} \quad k \to \infty.$$

As earlier (2.47) implies that $s_k \to \infty$ as $k \to \infty$. Arguing as in the proof of (i), consider a sequence $\tau_k \to \infty$ such that $I'_{12}(\eta_{12}(\tau_k, q^k)) \to 0$ and $a \leq I_{12}(\eta_{12}(\tau_k, q^k)) \leq b$. We will prove that $\eta_{12}(\tau_k, q^k) - \psi_{12}(v(q^k))$ has a convergent subsequence. Indeed Proposition 2.2' yields the existence of $(v_k) \subset \mathbf{R}^\ell$ such that

$$(2.48) \qquad (\eta_{12}(\tau_k, q^k)_i - v_k \quad \text{is convergent.}$$

Proposition 2.27' together with (2.45) then implies

$$(2.49) \qquad \sum_{i=1}^{2} [(\eta_{12}(\tau_k, q^k - \psi_{12}(v_k)))_i] = \sum_{i=1}^{2} [q_i^k] - 2v_k.$$

Thus the right hand side of (2.49) is convergent. By (2.44), $[q_1^k] + [q_2^k] - 2v(q^k)$ is bounded. Therefore it can be assumed that $v_k - v(q^k)$ is convergent. Hence $\eta_{12}(\tau_k, q^k) - \psi_{12}(v(q^k))$ has a convergent subsequence as stated. This shows, as in (i), the existence of γ and $M > 0$ such that if (2.40) holds with $q = q^k$ and $s \in [0, s_k]$, then

$$\|\eta_{12}(s, q^k) - \psi_{12}(v(q^k))\|_{W^{1,2}} \leq M.$$

Now (2.46) implies that $\|I'_{12}(\eta_{12}(s_k, q^k))\|_{W^{1,2}} > \gamma$ for large k. As in (i), the existence of τ_k implies $\overline{s}_k$, the largest positive $s \leq s_k$ such that $\|I'_{12}(\eta_{12}(\overline{s}_k, q^k))\|_{W^{1,2}} \leq \gamma$ is well defined. Then

$$(2.50) \qquad \|I'_{12}(\eta_{12}(s, q^k))\|_{W^{1,2}} \geq \gamma$$

for $s \in [s_k, \overline{s}_k]$ and

$$(2.51) \qquad \|\eta_{12}(\overline{s}_k, q^k) - \psi_{12}(v(q^k))\|_{W^{1,2}} \leq M.$$

By (2.47), we conclude as in (i) that

$$\|\eta_{12}(s_k, q^k) - \eta_{12}(\bar{s}_k, q^k)\|_{W^{1,2}} \le \int_{\bar{s}_k}^{s_k} \|I'_{12}(\eta_{12}(s, q^k))\|_{W^{1,2}} ds$$
$$\le \frac{1}{\gamma} \int_{\bar{s}_k}^{s_k} \|I'_{12}(\eta_{12}(s, q^k))\|^2_{W^{1,2}} ds \le \frac{b-a}{\gamma}. \tag{2.52}$$

Thus using (2.51)-(2.52), we have

$$\|\eta_{12}(s_k, q^k) - \psi_{12}(v(q^k))\|_{W^{1,2}} \le M + \frac{b-a}{\gamma}. \tag{2.53}$$

But (2.53) contradicts (2.46) and (ii) follows.

§3. A Morse Lemma near infinity

Proposition 2.2 describes the failure of the Palais-Smale condition for $I(q)$. Our main result in this section provides us with a kind of Morse Lemma for a suitable neighborhood of the set where (PS) fails. For the sake of simplicity this result is presented for the case where $\|q_1^k - q_2^k\|_{W^{1,2}}$ remains bounded while $\|\dot{q}_3^k\|_{L^2} \to 0$ and $|\left[q_3^k - \frac{q_1^k + q_2^k}{2}\right]| \to \infty$. Stated informally, we will show there is a neighborhood of "infinity" in which there is a change of variables $q = (q_1, q_2, q_3) \to (q_1, q_2, Q_3)$ such that

$$I(q) = I_{12}(q_1, q_2) + \frac{1}{2}\int_0^1 |\dot{Q}_3|^2 dt + \frac{1}{1 + |\left[Q_3 - \frac{q_1 + q_2}{2}\right]|^2}.$$

To state the result more precisely, let

$$\mathcal{Q} \equiv \left(1 + |\left[Q_3 - \frac{(q_1 + q_2)}{2}\right]|^2\right)^{-1}$$

and

$$\Psi(q) = \frac{1}{2}\int_0^1 |\dot{Q}_3|^2 dt + \mathcal{Q}.$$

Proposition 3.1:

1° Let V satisfy $(V_1) - (V_3)$. Given any $C > 0$, there exists an $\alpha(C) > 0$ such that whenever $q = (q_1, q_2, q_3) \in \Lambda$ satisfies

(i) $\|q_1 - v(q)\|_{L^\infty} + \|q_2 - v(q)\|_{L^\infty} \le C$ and

(ii) $\frac{1}{2}\|\dot{q}_3\|_{L^2}^2 + \dfrac{1}{1 + |[q_3] - v(q)|^2} \le \alpha(C)$

for some $v(q) \in \mathbf{R}^\ell$, then there exists a unique $\lambda(q) > 0$ and

$$Q_3 = \frac{[q_1 + q_2]}{2} + \frac{1}{\lambda}(q_3 - [q_3]) + \lambda\left(\left[q_3 - \frac{q_1 + q_2}{2}\right]\right) \tag{3.2}$$

such that

$$I(q) = I_{12}(q_1, q_2) + \Psi(q). \tag{3.3}$$

Moreover λ is differentiable.

2°. Conversely let V satisfy $(V_1) - (V_5)$. Given any $C > 0$, there exists an $\overline{\alpha}(C) > 0$ such that whenever $(q_1, q_2, Q_3) \in \Lambda$ satisfies

(iii) $\|q_1 - v(q)\|_{L^\infty} + \|q_2 - v(q)\|_{L^\infty} \le C$ and

(iv) $\frac{1}{2}\|\dot{Q}_3\|_{L^2}^2 + \dfrac{1}{1+|[Q_3]-v(q)|^2} \le \overline{\alpha}(C)$

for some $v(q) \in \mathbf{R}^\ell$, then there exists a unique $\mu(q_1, q_2, Q_3) > 0$ and

$$q_3 = \frac{[q_1+q_2]}{2} + \frac{1}{\mu}(Q_3 - [Q_3]) + \mu\left(\left[Q_3 - \frac{q_1+q_2}{2}\right]\right) \tag{3.4}$$

such that

$$I(q_1, q_2, q_3) = I_{12}(q_1, q_2) + \Psi(q). \tag{3.5}$$

Moreover $\mu(q_1, q_2, Q_3)$ is differentiable.

3°. If V satisfies $(V_1) - (V_5)$ and $\overline{\alpha}(C) = \alpha(C)$ are chosen still smaller, then $\lambda(q_1, q_2, q_3)\mu(q_1, q_2, Q_3) = 1$, and the transformations defined in 1° and 2° are inverse diffeomorphisms.

Remark 3.6: Conditions (i) -(ii) and (iii) - (iv) may be replaced by:

(v) $\sum_{i=1}^2 \|q_i - \dfrac{[q_1+q_2]}{2}\|_{L^\infty} \le C_1$ and

(vi) $\frac{1}{2}\|\dot{q}_3\|_{L^2}^2 + \dfrac{1}{1+|\left[q_3 - \frac{q_1+q_2}{2}\right]|^2} \le \beta(C_1)$

with q_3 replaced by Q_3 for (iv). Indeed if (q_1, q_2, q_3) satisfies (i)-(ii), then $|[q_i] - v(q)| \le C$ for $i = 1, 2$ and $\left|\dfrac{[q_1+q_2]}{2} - v(q)\right| \le C$. Therefore (i)-(ii) imply (v)-(vi) with $C_1 = 2C$ and $\beta(C_1)$ replaced by a suitable new constant. Obviously (v)-(vi) imply (i)-(ii) with $v(q) = \frac{1}{2}[q_1 + q_2]$.

Proof of Proposition 3.1: 1°. Let $C > 0$, q satisfy (i)-(ii), and set $w_3 = q_3 - [q_3]$. To verify (3.3) we must show if $\alpha(C)$ is small enough, the equation

$$\begin{aligned} &\frac{1}{2\lambda^2}\int_0^1 |\dot{w}_3|^2 dt + \frac{1}{1+\lambda^2|\left[q_3 - \frac{q_1+q_2}{2}\right]|^2} \\ &\qquad = I(q_1, q_2, q_3) - I_{12}(q_1, q_2) \end{aligned} \tag{3.7}$$

has a unique solution $\lambda > 0$. The function

$$\phi_q(\lambda) \equiv \frac{1}{2\lambda^2}\int_0^1 |\dot{w}_3|^2 dt + \frac{1}{1+\lambda^2|\left[q_3 - \dfrac{q_1+q_2}{2}\right]|^2}$$

is nonincreasing in λ. Clearly

$$\lim_{\lambda\to 0} \phi_q(\lambda) \ge 1 \tag{3.8}$$

and $\phi_q(\lambda)$ decreases to 0 as $\lambda \to \infty$ unless $[q_3] = \frac{1}{2}[q_1 + q_2]$. If $[q_3] = \frac{1}{2}[q_1 + q_2]$, then

$$|[q_3] - v(q)| \leq \frac{1}{2}\sum_{i=1}^{2} |[q_i] - v(q)| \leq \frac{C}{2}. \tag{3.9}$$

Thus

$$\frac{1}{2}\|\dot{q}_3\|_{L^2}^2 + \frac{1}{1 + |(q_3) - v(q)|^2} \geq \frac{4}{4 + C^2}. \tag{3.10}$$

Requiring that $\alpha(C) < 4(4 + C^2)^{-1}$ shows (3.10) does not hold and $\phi_q(\lambda)$ is strictly decreasing for $\lambda > 0$ and tends to 0 as $\lambda \to \infty$. Consequently (3.8) then implies that (3.7) has a unique solution if

$$I(q_1, q_2, q_3) - I_{12}(q_1, q_2) < 1. \tag{3.11}$$

Let $\tilde{V}_{ij}(x) = V_{ij}(x) + V_{ji}(-x)$ for $1 \leq i \neq j \leq 3$. Then (3.11) is equivalent to

$$\int_0^1 \left(\frac{1}{2}|\dot{q}_3|^2 - \tilde{V}_{13}(q_1 - q_3) - \tilde{V}_{23}(q_2 - q_3)\right) dt < 1. \tag{3.12}$$

By (i)-(ii),

$$\begin{aligned} |q_3 - q_i| &\geq |q_3 - v(q)| - |v(q) - q_i| \geq |[q_3] - v(q)| - |q_3 - [q_3]| - |v(q) - q_i| \\ &\geq |[q_3] - v(q)| - \|\dot{q}_3\|_{L^2} - |v(q) - q_i| \\ &\geq \left(\frac{1}{\alpha(C)} - 1\right)^{1/2} - (2\alpha(C))^{1/2} - C, \quad i = 1, 2. \end{aligned} \tag{3.13}$$

Thus (3.13) shows

$$\inf_{t \in [0,1]} |q_3(t) - q_i(t)| \to \infty$$

as $\alpha(C) \to 0$ for $i = 1, 2$. Consequently $(V_2) - (V_3)$ imply that if $\alpha(C)$ is small enough, we have

$$-\int_0^1 (\tilde{V}_{13}(q_1 - q_3) + \tilde{V}_{23}(q_2 - q_3))dt < \frac{1}{2} \tag{3.14}$$

for any q satisfying (i)-(ii). If we further require $\alpha(C) < \frac{1}{2}$, then by (ii),

$$\frac{1}{2}\int_0^1 |\dot{q}_3|^2 dt < \alpha(C) < \frac{1}{2} \tag{3.15}$$

and (3.12) is satisfied. Thus $1°$ is proved. Observe also that λ is differentiable since $V \in C^2$.

2°. Let $W_3 = Q_3 - [Q_3]$. Suppose $C > 0$ and (q_1, q_2, Q_3) satisfy (iii)-(iv). We want to show if $\alpha(C)$ is possibly still smaller than in 1°, the equation

(3.16)

$$\begin{aligned}\psi_Q(\mu) \equiv & \frac{1}{2\mu^2}\int_0^1 |\dot{W}_3|^2 dt - \int_0^1 \tilde{V}_{13}\left(\frac{[q_1+q_2]}{2} + \mu\left(\left[Q_3 - \frac{q_1+q_2}{2}\right]\right) + \frac{1}{\mu}W_3 - q_1\right)dt \\ & - \int_0^1 \tilde{V}_{23}\left(\frac{[q_1+q_2]}{2} + \mu\left(\left[Q_3 - \frac{q_1+q_2}{2}\right]\right) + \frac{1}{\mu}W_3 - q_2\right)dt \\ & = \Psi(q)\end{aligned}$$

has a unique solution $\mu > 0$. Clearly $\psi_Q(\mu)$ is well defined if

$$\frac{[q_1+q_2]}{2} + \mu\left(\left[Q_3 - \frac{q_1+q_2}{2}\right]\right) + \frac{1}{\mu}W_3(t) - q_i(t) \neq 0 \tag{3.17}$$

for all $t \in [0,1]$ and $i = 1, 2$. Assume (3.17) for the moment. Let $\overline{\mu}$ be any solution of (3.16). We consider the dependence of $\overline{\mu}$ on $\Psi(q)$. We claim

$$\lim_{\Psi(q)\to 0} \overline{\mu}\left|\left[Q_3 - \frac{q_1+q_2}{2}\right]\right| = \infty \tag{3.18}$$

and

$$\lim_{\Psi(q)\to 0} (\overline{\mu})^{-1}\|W_3\|_{L^\infty} = 0 \tag{3.19}$$

uniformly for (q_1, q_2, Q_3) satisfying (iii)-(iv). To prove (3.19), note that at any $\overline{\mu}$ satisfying (3.16), by (V_2) we have

$$\frac{1}{2\overline{\mu}^2}\int_0^1 |\dot{W}_3|^2 dt \leq \Psi(q). \tag{3.20}$$

Since $[W_3] = 0$,

$$\left\|\frac{W_3}{\overline{\mu}}\right\|_{L^\infty} \leq \left\|\frac{\dot{W}_3}{\overline{\mu}}\right\|_{L^2} \leq (2\Psi(q))^{1/2} \tag{3.21}$$

via (3.20) which yields (3.19). To get (3.18), note that by (iii) and (3.21),

$$\begin{aligned}&\left|\frac{[q_1+q_2]}{2} + \overline{\mu}\left(\left[Q_3 - \frac{q_1+q_2}{2}\right]\right) + \frac{1}{\overline{\mu}}W_3(t) - q_i(t)\right| \\ &\quad \leq \frac{\|[q_1] - v(q)\|}{2} + \frac{\|[q_2] - v(q)\|}{2} + \|q_i - v(q)\|_{L^\infty} + \overline{\mu}\left|\left[Q_3 - \frac{q_1+q_2}{2}\right]\right| \\ &\quad + \left\|\frac{W_3}{\overline{\mu}}\right\|_{L^\infty} \leq \frac{3C}{2} + (2\Psi(q))^{1/2} + \overline{\mu}\left|\left[Q_3 - \frac{q_1+q_2}{2}\right]\right|\end{aligned} \tag{3.22}$$

for $i = 1, 2$. If there were a sequence (q_1^k, q_2^k, Q_3^k) for which $\overline{\mu}\left|\left[Q_3^k - \frac{q_1^k + q_2^k}{2}\right]\right|$ remained bounded while $\Psi(q^k) \to 0$, then

$$-\int_0^1 \tilde{V}_{13}\left(\frac{[q_1^k + q_2^k]}{2} + \overline{\mu}_k\left(\left[Q_3^k - \frac{q_1^k + q_2^k}{2}\right]\right) + \frac{1}{\overline{\mu}_k} W_3^k - q_i^k\right) dt$$

would be bounded from below by a positive number uniformly in k as a consequence of $(V_1) - (V_3)$, and (3.22). This would contradict (3.16). Therefore (3.18) holds for any solution $\overline{\mu}$.

Next observe that $\Psi(q) \to 0$ if and only if

$$\frac{1}{2}\|\dot{Q}_3\|_{L^2}^2 + \frac{1}{1 + |[Q_3] - v(q)|^2} \to 0$$

for any (q_1, q_2, Q_3) satisfying (iii). Indeed we have

$$\begin{aligned} |[Q_3] - v(q)| &\le \left|\left[Q_3 - \frac{q_1 + q_2}{2}\right]\right| + \frac{1}{2}\sum_{i=1}^{2} |[q_i] - v(q)| \\ &\le \left|\left[Q_3 - \frac{q_1 + q_2}{2}\right]\right| + \frac{C}{2} \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} \left|\left[Q_3 - \frac{q_1 + q_2}{2}\right]\right| &\le |[Q_3] - v(q)| + \frac{1}{2}\sum_{i=1}^{2} |[q_i] - v(q)| \\ &\le |[Q_3] - v(q)| + \frac{C}{2}. \end{aligned} \tag{3.24}$$

Thus $\Psi(q)$ is small is equivalent to $\overline{\alpha}(C)$ is small since (q_1, q_2, Q_3) satisfies (iii) and (iv). We require that

$$\overline{\alpha}(C) < \frac{4}{4 + C^2}.$$

Then $\left|\left[Q_3 - \frac{q_1 + q_2}{2}\right]\right|$ is strictly positive by the argument following (3.10). Therefore

$$\mu\left|\left[Q_3 - \frac{q_1 + q_2}{2}\right]\right| \to \infty$$

as $\mu \to \infty$. Since

$$\left\|\frac{W_3}{\mu}\right\|_{L^\infty} + \left\|\frac{[q_1 + q_2]}{2} - q_i\right\|_{L^\infty}$$

remains bounded, $i = 1, 2$, as $\mu \to \infty$, we see from (V_3) that

$$(3.25) \qquad \psi_Q(\mu) \text{ is defined for large } \mu \text{ and } \lim_{\mu\to\infty} \psi_Q(\mu) = 0.$$

The interval on which $\psi_Q(\mu)$ is defined can be characterized further. By (iii), for $i = 1, 2$,

$$\left\| \frac{[q_1 + q_2]}{2} - q_i \right\|_{L^\infty} \le \frac{3C}{2}.$$

Using (3.21), $\psi_Q(\mu)$ is defined if

$$(3.26) \qquad \mu \left| \left[Q_3 - \frac{q_1 + q_2}{2} \right] \right| \ge \frac{3C}{2} + \frac{\|\dot{W}_3\|_{L^2}}{\mu} + 1.$$

As noted above, $\left[Q_3 - \dfrac{q_1 + q_2}{2} \right]$ is nonzero. Hence (3.26) defines an interval $[\mu_1, \infty]$ since if (3.26) holds for some μ, it holds for any $\hat{\mu} > \mu$. Let us compare $\psi_Q(\mu_1)$ and $\Psi(q)$. Either

(a) $\dfrac{1}{2\mu_1^2}\|\dot{W}_3\|_{L^2}^2 > \Psi(q)$

and then by (3.16) and (V_2), $\psi_Q(\mu_1) > \Psi(q)$ or

(b) $\dfrac{1}{2\mu_1^2}\|\dot{W}_3\|_{L^2}^2 \le \Psi(q)$

in which case

$$(3.27) \qquad \mu_1 \left| \left[Q_3 - \frac{Q_1 + q_2}{2} \right] \right| = \frac{3C}{2} + 1 + \frac{\|\dot{W}_3\|_{L^2}}{\mu_1} \le \frac{3C}{2} + 1 + (2\Psi(q))^{1/2}$$

and for $i = 1, 2$,

$$\begin{aligned}(3.28) \left| \frac{[q_1 + q_2]}{2} + \mu_1 \left(\left[Q_3 - \frac{q_1 + q_2}{2} \right] \right) + \frac{1}{\mu_1} W_3(t) - q_i(t) \right| \\ \le \sum_{j=1}^{2} \frac{|[q_j] - v(q)|}{2} + |v(q) - q_i(t)| + \frac{\|\dot{W}_3\|_{L^2}}{\mu_1} + \mu_1 \left| \left[Q_3 - \frac{q_1 + q_2}{2} \right] \right| \\ \le 3C + 1 + (2\Psi(q))^{1/2}.\end{aligned}$$

By the above remarks, $\overline{\alpha}(C)$ small implies $\Psi(q) < \frac{1}{2}$. Hence there is an α_0 such that if $\overline{\alpha}(C) < \alpha_0$ and $i = 1, 2$,

$$(3.29) \qquad \left| \frac{[q_1 + q_2]}{2} + \mu_1 \left(\left[Q_3 - \frac{q_1 + q_2}{2} \right] \right) + \frac{W_3(t)}{\mu_1} - q_i(t) \right| \le 3 + 3C$$

for all $t \in [0, 1]$. Now (3.29), (3.16), and (V_2) imply the existence of a constant $\beta(C) > 0$ such that

$$(3.30) \qquad \psi_Q(\mu_1) \ge \beta(C)$$

where $\beta(C)$ is independent of (q_1, q_2, Q_3) satisfying (iii)-(iv) (provided that $\overline{\alpha}(C) < \alpha_0$). If we further choose $\alpha(C)$ so small, say $\overline{\alpha}(C) < \alpha_1$, so that

$$\Psi(q) < \beta(C), \tag{3.31}$$

(3.30)-(3.31) show $\psi_Q(\mu_1) > \Psi(q)$ for case (b) as well as for case (a). This coupled with (3.23) shows that (3.16) has a solution $\overline{\mu}$. Observe that any solution satisfies (3.18)-(3.19). Hence (3.26) follows from (3.18)-(3.19) for $\mu = \overline{\mu}$ provided that $\Psi(q)$ is small enough. Therefore $\mu \in [\mu_1, \infty)$ if $\alpha(C)$ is small enough.

To prove the uniqueness and differentiability of $\overline{\mu}$, we need only show

$$\psi'_Q(\overline{\mu}) < 0. \tag{3.32}$$

For $i = 1, 2$, let

$$x_i \equiv \frac{[q_1 + q_2]}{2} + \overline{\mu}\left(\left[Q_3 - \frac{q_1 + q_2}{2}\right]\right) + \overline{\mu}^{-1} W_3 - q_i,$$

$$y = \left[Q_3 - \frac{q_1 + q_2}{2}\right] - \overline{\mu}^{-2} W_3,$$

and

$$z_i = -2\overline{\mu}^{-1} W_3 + q_i - \frac{[q_1 + q_2]}{2}.$$

Then

$$\begin{aligned} \psi'_Q(\overline{\mu}) &= -\overline{\mu}^{-3} \int_0^1 |\dot{W}_3|^2 dt - \int_0^1 V'_{13}(x_1) \cdot y dt \\ &\quad + \int_0^1 V'_{31}(-x_1) \cdot (y) dt - \int_0^1 V'_{23}(x_2) \cdot y dt + \int_0^1 V'_{32}(-x_2) \cdot y dt \\ &= -\overline{\mu}^{-3} \int_0^1 |\dot{W}_3|^2 dt - \overline{\mu}^{-1} \int_0^1 V'_{13}(x_1) \cdot (x_1 + z_1) dt \\ &\quad + \int_0^1 V'_{31}(-x_1) \cdot (x_1 + z_1) - V'_{23}(x_2) \cdot (x_2 + z_2) \\ &\quad + V'_{32}(-x_2) \cdot (x_2 + z_2) dt. \end{aligned} \tag{3.33}$$

From (3.18)-(3.19) and (iii), we know that for $i = 1, 2$,

$$|x_i(t)| \geq \overline{\mu}\left|\left[Q_3 - \frac{q_1 + q_2}{2}\right]\right| - \left\|\frac{W_3}{\overline{\mu}}\right\|_{L^\infty} - \frac{3}{2}C \to \infty \tag{3.34}$$

as $\Psi(q) \to 0$ and

$$\|z_i(t)\|_{L^\infty} \leq \frac{3C}{2} + \frac{2\|W_3\|_{L^\infty}}{\overline{\mu}} \leq \frac{3C}{2} + 1 \tag{3.35}$$

if $\overline{\alpha}(C)$ is small enough. Using (V_5), there is an α_2 such that if $\overline{\alpha}(C) < \alpha_2$,

$$(3.36)\qquad V'_{3i}(x_i)\cdot x_i - \left(\frac{3C}{2}+1\right)|V'_{3i}(x_i)| > 0$$

$$-V'_{i3}(-x_i)\cdot x_i - \left(\frac{3C}{2}+1\right)|V'_{i3}(-x_i)| > 0$$

for $i = 1,2$. By (3.36), $\psi'_Q(\overline{\mu}) < 0$ if $\overline{\alpha}(C) < \alpha_2$ and the proof of 2° is complete.

Lastly to prove 3°, let q_3 be defined by (3.4). Then

$$(3.37)\qquad [q_3] = \left[\frac{q_1+q_2}{2}\right] + \mu\left[Q_3 - \frac{q_1+q_2}{2}\right]$$

and

$$(3.38)\qquad q_3 - [q_3] = \mu^{-1}(Q_3 - [Q_3]).$$

Solving (3.37)-(3.38) for $[Q_3]$, $Q_3 - [Q_3]$ yields:

$$(3.39)\qquad [Q_3] = \left[\frac{q_1+q_2}{2}\right] + \mu^{-1}\left[q_3 - \frac{q_1+q_2}{2}\right]$$

and

$$(3.40)\qquad Q_3 - [Q_3] = \mu(q_3 - [q_3]).$$

In the proof of 2°, it was shown that $\Psi(q) \to 0$ is equivalent to $\overline{\alpha}(C) \to 0$. Consequently, recalling that $W_3 = Q_3 - [Q_3]$, (3.39)-(3.40), (3.18), and (3.21) show that $\left|\left[q_3 - \frac{q_1+q_2}{2}\right]\right| \to \infty$ and $\|\dot{q}_3\|_{L^2} \to 0$ as $\overline{\alpha} \to 0$. Hence (q_1, q_2, q_3) satisfies (i) and (ii) for $\overline{\alpha}$ small. Therefore (3.2) holds. Comparing (3.2) to (3.39)-(3.40) shows $\lambda(q_1, q_2, q_3) = \mu^{-1}(q_1, q_2, Q_3)$. The composite of the transformations of 1° and 2° is then readily seen to be the identity and 3° easily follows.

In §5, we will need the following consequence of Proposition 3.1:

Corollary 3.41: Let $(q_1, q_2) \in \Lambda_{12}$ and set

$$C_2(q_1, q_2) = \sum_{i=1}^{2} \|q_i - \frac{1}{2}[q_1 + q_2]\|_{L^\infty}.$$

Then there exists a continuous function $\alpha(q_1, q_2)$ such that if q_3 satisfies (ii) of Proposition 3.1 with $\alpha(q_1, q_2)$, then both systems of coordinates given by Proposition 3.1 are available at (q_1, q_2, q_3).

Proof: By Proposition 3.1, for any $(q_1, q_2) \in \Lambda_{12}$ and $C = C_2(q_1, q_2)$, there is an $\alpha(2C)$ for which the conclusions of the Proposition are valid at (q_1, q_2, q_3) for any q_3 satisfying (ii) (with $v = \frac{1}{2}[q_1 + q_2]$). Note that (i) is satisfied with C replaced by $2C$ for all points in a neighborhood W_{q_1, q_2} of (q_1, q_2). This gives us a covering of Λ_{12} which possesses a locally finite refinement $\{W_m\}$. Let (ρ_m) be a smooth partition of unity subordinate to $\{W_m\}$. Now define

$$\alpha(q_1, q_2) = \sum_m \alpha(2C)_{W_m} \rho_m(q_1, q_2) \tag{3.42}$$

where $\alpha(2C)_{W_m}$ is the $\alpha(2C)$ associated to some W_{q_1, q_2} such that $W_m \subset W_{q_1, q_2}$. Then by (3.42),

$$\alpha(q_1, q_2) \le \alpha(C)_W$$

where $\alpha(C)_W$ is the largest of the α_{W_m} such that $\rho_{W_m}(q_1, q_2) \neq 0$. Since Proposition 3.1 holds at (q_1, q_2, q_3) where q_3 satisfies (ii) with $\alpha = \alpha(C)_W$, it holds a fortiori for a subclass of q_3's with $\alpha = \alpha(q_1, q_2)$.

Remark 3.43: The function $\alpha(q_1, q_2)$ may be chosen so that it is differentiable with derivative bounded by $\bar{\epsilon}$. Namely we can take ρ_W satisfying $|\rho_W'| \le K_W \rho$ for a suitable constant K_W and then choose $\alpha(C)_W < K_W^{-1} \bar{\epsilon}$.

We conclude this section with two more corollaries of Proposition 3.1. The first of these provides, modulo translations, a priori bounds for critical points of I which depend on the corresponding critical value.

Corollary 3.44: Let V satisfy $(V_1) - (V_5)$. Let $M \ge \epsilon_1 > 0$ be given. Then there exists a constant $C(\epsilon_1, M) > 0$ such that for any solution $q = (q_1, q_2, q_3)$ of (HS) satisfying $\epsilon_1 \le I(q) \le M$, there is a $v(q) \in \mathbf{R}^\ell$ so that

$$\sum_{i=1}^{3} \|q_i - v(q)\|_{W^{1,2}} \le C(\epsilon_1, M).$$

Proof: If the Corollary is false, there is a sequence $(q^k) = (q_1^k, q_2^k, q_3^k)$ such that $I(q^k) \to c \in [\epsilon_1, M]$, $I'(q^k) \to 0$, and $q^k - \psi(\xi_k)$ is not bounded in $W^{1,2}$ for $(\xi_k) \subset \mathbf{R}^\ell$. Since $(I(q^k)) \subset [\epsilon_1, M]$, $\dot{q}^k$ is bounded in L^2. Therefore $([q^k] - \psi(\xi_k))$ is not bounded in $\mathbf{R}^{3\ell}$. An argument as in the proof of Proposition 2.2 shows $[q_i^k - q_j^k]$ converges (along a subsequence) for some $i \neq j$ for otherwise $I(q^k) \to 0$. Hence without loss of generality, we can assume $[q_1^k - q_2^k]$ converges. Let $\xi_k = \frac{1}{2}[q_1^k + q_2^k]$. Then $[q_i^k] - \xi_k$ is bounded, $i = 1, 2$ and we can assume $q_i^k - \xi_k$ converges, $i = 1, 2$. As in the proof of Proposition 2.2 again,

this implies $V'_{13}(q_1^k - q_3^k)$, $V'_{23}(q_2^k - q_3^k)$, etc. $\to 0$ in L^∞. Hence by (HS), $\ddot{q}_3^k \to 0$ in L^∞ so $\dot{q}_3^k \to 0$ in L^2. Choose C so that q^k satisfies (i) of Proposition 3.1. Then for large k, q^k also satisfies (ii) of Proposition 3.1. For such k, we may write

$$I(q^k) = I_{12}(q_1^k, q_2^k) + \Psi(q^k). \tag{3.45}$$

Therefore by Remark 3.6, (q_1^k, q_2^k, Q_3^k) also satisfies (iii) and (iv) of Proposition 3.1 for large k. By $2°(b)$ of Proposition 2.2, $I(q^k) - I_{12}(q_1^k, q_2^k) \to 0$. Consequently the sequence enters the domain where the map

$$\phi : (q_1, q_2, q_3) \to (q_1, q_2, Q_3) \tag{3.46}$$

is a diffeomorphism. In this domain, critical points of $I(q)$ are also critical points of

$$\mathcal{I}(q_1, q_2, Q_3) \equiv I_{12}(q_1, q_2) + \Psi(q) \tag{3.47}$$

in the (q_1, q_2, Q_3) coordinates. At such a critical point, $[Q_3] = \frac{1}{2}[q_1 + q_2]$. As has been noted earlier, $\alpha(C) < 4(4 + C^2)^{-1}$ implies this is impossible. Thus $\mathcal{I}$ has no critical points in this region and there does not exist a sequence as above. The Corollary is proved.

The final result in this section shows that the only "(PS) sequences" in a neighborhood of infinity are those which have a "two body" limit.

Corollary 3.48: Let V satisfy $(V_1) - (V_6)$. If $C > 0$ and $0 < a < b$, then there exists an $\alpha(C, a, b)$ such that whenever (q^k) is a sequence in Λ satisfying (i) and (ii) of Proposition 3.1 with C and $\alpha(C, a, b)$ and such that $I(q^k) \to c \in [a, b]$ and $I'(q^k) \to 0$, then $I_{12}(q_1^k, q_2^k) \to c$, $I'_{12}(q_1^k, q_2^k) \to 0$, and $\Psi(q^k) \to 0$. In particular (iii) and (iv) of Proposition 3.1 are satisfied by (q_1^k, q_2^k, Q_3^k) for large k.

Conversely let (q_1^k, q_2^k, Q_3^k) be a sequence in Λ satisfying (iii) and (iv) of Proposition 3.1 and such that $I_{12}(q_1^k, q_2^k) \to c \in [a, b]$, $I'_{12}(q_1^k, q_2^k) \to 0$, and $\Psi(q^k) \to 0$. Then $I(q_1^k, q_2^k, q_3^k) \to c$ and $I'(q_1^k, q_2^k, q_3^k) \to 0$. In particular (i) and (ii) of Proposition 3.1 are satisfied by (q_1^k, q_2^k, q_3^k) for large k.

Proof: Suppose (q^k) satisfies (i) and (ii) of Proposition 3.1 and $I(q^k) \to c$, $I'(q^k) \to 0$. Then either 1° or 2° of Proposition 2.2 holds. If 2° holds, there is a subsequence of q^k and a pair of indices $r \neq j \in \{1, 2, 3\}$ such that $\|[q_r^k - q_j^k]\|$ is bounded and $\|[q_i^k - q_r^k]\|$, $\|[q_i^k - q_j^k]\| \to \infty$ where i is the third index. Then by (i) and Remark 3.6,

$$\sum_{\ell=1}^{2} \|q_\ell^k - \frac{1}{2}[q_1^k + q_2^k]\|_{L^\infty} \le C_1. \tag{3.49}$$

Thus (3.49) shows $|[q_1^k - q_2^k]|$ is bounded so 2° holds, $\{r,j\} = \{1,2\}$ and $i = 3$. Hence by 2°, $\|\dot{q}_3^k\|_{L^2} \to 0$, $|[q_3^k - \frac{1}{2}(q_1^k + q_2^k)]| \to \infty$, $I_{12}(q_1^k, q_2^k) \to c$ and $I'_{12}(q_1^k, q_2^k) \to 0$. Thus $\Psi(q^k) \to 0$ along this subsequence. Therefore (iii) and (iv) hold for large k along this subsequence.

If 2° of Proposition 2.2 does not hold along a subsequence, $|[q_i^k - q_j^k]|$ is bounded for $i \neq j \in \{1,2,3\}$. We can take $v_k = \frac{1}{2}[q_1^k + q_2^k]$ as in (2.7). By 1° of Proposition 2.2, $q_i^k - v_k \to q^\infty$ in $W^{1,2}$ for $i = 1,2,3$. By (3.49),

$$\|q_1^\infty\|_{L^\infty} + \|q_2^\infty\|_{L^\infty} \le C_1. \tag{3.50}$$

By Corollary 3.44, there is a $w(q^\infty) \in \mathbf{R}^\ell$ such that

$$\sum_{i=1}^{3} \|q_i^\infty - w(q^\infty)\|_{W^{1,2}} \le C(a,b). \tag{3.51}$$

Therefore by (3.50)-(3.51),

$$|w(q^\infty)| \le C_1 + C(a,b). \tag{3.52}$$

Now by (ii) and Remark 3.6,

$$\begin{aligned} \sqrt{\frac{1}{\beta(C_1)} - 1} &\le |[q_3^k] - v(q_3^k)| \\ &\le |[q_3^k] - v(q_3^k) - [q_3^\infty]| + |[q_3^\infty] - w(q^\infty| + |w(q^\infty)|. \end{aligned} \tag{3.53}$$

Hence for large k, by (3.51)-(3.53),

$$\sqrt{\frac{1}{\beta(C_1)} - 1} \le 1 + 2C(a,b) + C_1. \tag{3.54}$$

But as $\alpha(C) \to 0$, the left-hand side of (3.54) $\to \infty$ while the right-hand side remains bounded. Thus (3.54) cannot hold for small α and we must be in case 2° of Proposition 2.2 along our subsequence. Finally observing that what has just been established holds for a subsequence of any sequence satisfying $I(q^k) \to c$, $I'(q^k) \to 0$, our conclusion must hold for the entire sequence, and the first half of Corollary 3.48 is proved.

For the converse, suppose that (q_1^k, q_2^k, Q_3^k) is such that $I_{12}(q_1^k, q_2^k) \to c$, $I'_{12}(q_1^k, q_2^k) \to 0$, and $\Psi(q^k) \to 0$. Then the associated (q_1^k, q_2^k, q_3^k) satisfy:

$$I(q_1^k, q_2^k, q_3^k) = I_{12}(q_1^k, q_3^k) + \Psi(q^k). \tag{3.55}$$

Therefore $I(q_1^k, q_2^k, q_3^k) \to c$ and (3.55), $(V_2), (V_3)$ imply

$$\|\dot{q}_k\|_{L^2} \to 0 \quad \text{and} \quad -\int_0^1 V_{i3}(q_i^k - q_3^k)dt \to 0, \tag{3.56}$$

$i = 1, 2$. Proposition 2.2′ implies the existence of v_k such that $(q_k^i - v_k)$ converges for $i = 1, 2$ along a subsequence. Thus (i) of Proposition 3.1 holds for large k along this subsequence. Also (3.56) shows that $|[q_3^k] - v_k| \to \infty$ as $k \to \infty$ and $\|q_3^k - [q_3^k]\|_{W^{1,2}} \to 0$. Consequently $I'(q_1^k, q_2^k, q_3^k) \to 0$ and (ii) of Proposition 3.1 is satisfied for large k along our subsequence. As in the first part of this corollary, it then follows for the entire sequence and the proof is complete.

§4 A modified functional

Let $0 < \epsilon_1 < M$ with ϵ_1 small and M large. To prove Theorem 1, we would like to use the unstable manifolds for the negative gradient flow of I corresponding to critical points of I in I^M. Unfortunately the critical points of I might be degenerate and the gradient flow does not satisfy the (PS) condition so we cannot do this. Therefore we will approximate I by a new functional which is well behaved enough to permit the above ideas to work.

To help handle the fact that the critical points of I might be degenerate, we use Corollary 3.44 which yields the existence of a constant $C_1(\epsilon_1, M)$ such that for any critical point q of I satisfying $\frac{\epsilon_1}{2} \le I(q) \le M+1$, we have

$$\sum_{i=1}^{3} \|q_i - v(q)\|_{W^{1,2}} \le C_1\left(\frac{\epsilon_1}{2}, M+1\right) \tag{4.1}$$

for suitable $v(q) \in \mathbf{R}^\ell$. Since our functional is invariant under translations in the sense described in (2.24), (4.1) and (HS) show the critical set of I in $I^{-1}\left(\frac{\epsilon_1}{2}, M+1\right)$ is compact after quotienting out the translations and I' on this quotient space, $\tilde{\Lambda}$, is Fredholm and proper in a neighborhood, N, of this critical set. By (4.1) and Proposition 2.9, we have

Proposition 4.2: Let V satisfy $(V_1)-(V_5)$. Then for any $\overline{\delta} > 0$, there exists a functional $J \in C^2(\Lambda, \mathbf{R})$ such that

1° J is invariant under translations in the sense of (2.4),
2° $J = I$ in $\tilde{\Lambda}\backslash N_1$ where N_1 is a small neighborhood of N
3° $\|J - I\|_{C^2(\tilde{\Lambda},\mathbf{R})} \le \overline{\delta}$
4° $J\,|_{\tilde{\Lambda}}$ has only finitely many critical points in N_1
5° All critical points of $J\,|_{\tilde{\Lambda}}$ are nondegenerate and have finite Morse index
6° $J\,|_{\tilde{\Lambda}}$ satisfies (PS) for sequences in N_1
7° If $\overline{\delta}$ is sufficiently small, if $\frac{\epsilon_1}{2} \le J(q) \le M+1$ and $J'(q) = 0$, then for a suitable $v(q) \in \mathbf{R}^\ell$,

$$\sum_{i=1}^{3} \|q_i - v(q)\|_{W^{1,2}} < 1 + C_1\left(\frac{\epsilon_1}{2}, M+1\right)$$

8° $J' \neq 0$ on $I^{\epsilon_1} = J^{\epsilon_1}$ for ϵ_1 sufficiently small.

Proof: The result follows in a straightforward way from approximation arguments due to Marino and Prodi [7] and extended by Bahri [8] and Bahri-Berestycki [9].

Proposition 4.2 allows us to avoid problems of degeneracy for critical points of I in I^{M+1}. We apply the above procedure to each "2-body problem" associated with I. For

the sake of simplicity, we restrict our presentation to I_{12}. Let $K_{ij}(a,b)$ denote the set of critical points of I_{ij} with critical values between a and b. For $\rho > 0$, let $N(\rho)$ be a uniform ρ neighborhood of $K_{12}(\epsilon_1, M+1)$. Then taking the quotient $\tilde{\Lambda}_{12}$ of Λ_{12} by the translational symmetry, I'_{12} is Fredholm and proper on the image $\tilde{N}(\rho)$ of $N(\rho)$. Therefore as in Proposition 4.2, we may replace I_{12} by a new functional, J_{12}, which on $\tilde{\Lambda}_{12}$ between the levels $\epsilon_1/2$ and $M+1$ has only finitely many critical points. These points are also nondegenerate and have finite Morse index. We can also assume J_{12} is invariant under translations and $J_{12}\mid_{\tilde{\Lambda}_{12}} = I_{12}\mid_{\tilde{\Lambda}_{12}}$ outside $\tilde{N}(\rho)$. Moreover we may choose J_{12} as close as we want to I_{12} in the C^2 norm. In particular for all $\overline{\delta} > 0$, there is a J_{12} having the properties stated above and

$$(4.3) \qquad \|J_{12} - I_{12}\|_{C^2(\Lambda_{12},\mathbf{R})} < \overline{\delta}.$$

Next we suitably modify J using the functionals J_{ij} just constructed. Proposition 2.36(ii) provides us with a constant $\tilde{C}\left(\frac{\epsilon_1}{2}, M+1\right)$ such that for any $q \in N(\rho)$, there exists $v(q) \in \mathbf{R}^\ell$ satisfying

$$(4.4) \qquad \|(\eta_{ij}(s,q))_r - v(q)\|_{W^{1,2}} \le \tilde{C}, \quad r = 1,2$$

for any $s \ge 0$ such that

$$(4.5) \qquad \frac{\epsilon_1}{2} \le I_{ij}(\eta_{ij}(s,q)) \le M+1.$$

For future reference, observe that (4.4) holds for J_{ij} and the corresponding $\tilde{\eta}_{ij}$ with ϵ_1 and $M+\frac{1}{2}$ instead of $\epsilon_1/2$ and $M+1$. Indeed either $\tilde{\eta}_{ij}(s,q) \in N(\rho)$, in which case applying (4.4)-(4.5) with $s=0$, $q = \tilde{\eta}_{ij}(s,q)$ and $\frac{\epsilon_1}{2} \ge \overline{\delta}$ of (4.3), we derive the conclusion, or $\tilde{\eta}_{ij}(s,q) \notin N(\rho)$. For this latter case, let s_1 be the maximal time smaller than s such that $\tilde{\eta}_{ij}(s_1,q) \in N(\rho)$. Since $J_{ij}(x) = I_{ij}(x)$ outside $N(\rho)$, $\tilde{\eta}_{ij}(s,q) = \eta_{ij}(s-s_1, \tilde{\eta}_{ij}(s_1,q))$ and the same conclusion holds. Therefore for future use we have: for all $q \in N(\rho)$ which is a neighborhood of the critical set for J_{12} between the levels ϵ_1 and $M+1$, there exists $v(q) \in \mathbf{R}^\ell$ satisfying

$$(4.6) \qquad \|(\tilde{\eta}_{ij}(s,q))_r - v(q)\|_{W^{1,2}} \le \tilde{C}, \quad r = 1,2$$

for any $s \ge 0$ such that $J_{ij}(\tilde{\eta}_{ij}(s,q)) \ge \epsilon_1$.

As for (i)-(ii) and (iii)-(iv) of Proposition 3.1 for I - see Remark 3.6 - we can replace the conditions (4.4) and (4.6) by

$$(4.7) \qquad \sum_{i=1}^{2} \|(\eta_{ij}(s,q))_r - \frac{1}{2}[(\eta_{ij}(s,q))_1 + (\eta_{ij}(s,q))_2]\|_{W^{1,2}} \le 3\tilde{C}$$

and by the same expression with $\tilde{\eta}_{ij}$ instead of η_{ij}. Conversely a condition of the form (4.7) implies a condition like (4.4), (4.6) with e.g.

$$v(q) = \frac{1}{2}[(\eta_{ij}(s,q))_1 + (\eta_{ij}(s,q))_2]$$

for (4.4).

With $\tilde{C}$ as in (4.4) and (4.6), we now define

$$C = 6\tilde{C}. \tag{4.8}$$

With this choice of C in Proposition 3.1, there is a corresponding C_1 and $\beta(C_1)$ given by Remark 3.6 such that if (q_1, q_2, q_3) satisfy (v)-(vi) of Remark 3.6, then Proposition 3.1 holds. We choose C_1 still larger and $\beta(C_1)$ smaller so that in fact Proposition 3.1 applies for this choice of C_1, $\beta(C_1)$, both in the (q_1, q_2, q_3) and (q_1, q_2, Q_3) coordinates. Actually we will be using this fact more for the (q_1, q_2, Q_3) coordinates. We also further restrict C_1 and $\beta(C_1)$ so that Corollary 3.48 applies. If $\beta(C_1)$ is chosen still smaller, the three neighborhoods defined by (v)-(vi) of Remark 3.6 in the (q_i, q_j, Q_r) coordinates are pairwise disjoint and do not intersect the set

$$\left\{ (q_1, q_2, q_3) \mid \sum_{i=1}^{3} \|q_i - v(q)\|_{W^{1,2}} < 1 + C_1(\epsilon_1, M) \text{ for a suitable } v(q) \right\}$$

The following construction should be understood as being carried out with a permutation of indices. Let

$$\omega_{12} : \Lambda \longrightarrow [0,1] \tag{4.10}$$

be a C^∞ function such that $\omega_{12} = 1$ on $\mathcal{V}_1$, the set of (q_1, q_2, Q_3) satisfying (v)-(vi) (of Remark 3.6) with constants $C_1' = C_1/2$ and $\beta'(C_1) = \beta(C_1)/2$, and $\omega_{12} = 0$ outside of $\mathcal{V}_2$, the set of (q_1, q_2, Q_3) satisfying (v)-(vi) (with constants C_1 and $\beta(C_1)$). Note that $\mathcal{V}_1, \mathcal{V}_2 \subset I^{M+1}$.

We define a new functional $\tilde{I}$ as follows:

$$\tilde{I}(q) = \left(1 - \sum_{i<j} \omega_{ij}(q)\right) J(q) \tag{4.11}$$

$$+ \sum_{i<j} \omega_{ij}(q) \left(J_{ij}(q_i, q_j) + \frac{1}{2}\int_0^1 |Q_r|^2 dt + \frac{1}{1 + \|Q_r - \frac{q_i + q_j}{2}\|^2} \right)$$

where $r \in \{1,2,3\} \backslash \{i,j\}$. Defining $\mathcal{V}_1(i,j)$, etc. in the natural way, observe that in each neighborhood of type $\mathcal{V}_1(i,j)$, the functional

$$\tilde{I}(q) = J_{ij}(q_i, q_j) + \frac{1}{2} \int_0^1 |\dot{Q}_r|^2 dt + \frac{1}{1 + \|[Q_r - \frac{1}{2}(q_i + q_j)]\|^2}$$

and in each neighborhood of type $\mathcal{V}_2(i,j)$,

$$\tilde{I}(q) = \omega_{ij}(q)\left(J_{ij}(q_i, q_j) + \frac{1}{2} \int_0^1 |\dot{Q}_r|^2 dt + \frac{1}{1 + \|[Q_r - \frac{1}{2}(q_i + q_j)]\|^2}\right) + (1 - \omega_{ij}(q))J(q) \tag{4.12}$$

due to the fact that the sets $\mathcal{V}_2(i,j)$ are pairwise disjoint. Observe also that outside of the sets $\mathcal{V}_2(i,j)$, $\tilde{I}(q) = J(q)$. In particular, by the choice of $\beta(C_1)$, $\tilde{I} = J$ near the critical points of I having critical values between ϵ_1 and $M+1$ since these points satisfy (4.9). Therefore (4.11) does not change our previous modification of I near the critical set of I between $\epsilon_1/2$ and $M+1$.

Note that for any $\epsilon > 0$, we may choose the functionals J_{ij} so that

$$\|\tilde{I} - J\|_{C^2} < \epsilon. \tag{4.13}$$

Indeed from (4.12) and (3.5),

$$\|\tilde{I} - J\|_{C^2} \leq \|\Sigma \omega_{ij}(J_{ij} - I_{ij} + I - J)\|_{C^2}. \tag{4.14}$$

Since the ω_{ij}'s are fixed, for $\bar{\delta}$ sufficiently small, (4.14), 3° of Proposition 4.2, and (4.3) imply (4.13).

Our next step involves the definition of a suitable pseudogradient vector field, $\tilde{Z}$, for $\tilde{I}$. For the sake of simplicity, we consider the case $i = 1$, $j = 2$; the other cases are obtained in the same way. Let

$$\mathcal{V}_0 = \{(q_1, q_2, Q_3) \in \Lambda \mid (q_1, q_2, Q_3) \text{ satisfies (4.15)-(4.16)}\} \cap I^{M+1}$$

where

$$\sum_{i=1}^{2} \left\| q_i - \left[\frac{q_1 + q_2}{2}\right] \right\|_{L^\infty} \leq C_1/4 \tag{4.15}$$

$$\Psi(q) \leq \beta_1 \tag{4.16}$$

where $\beta_1 \leq \frac{\beta(C_1)}{4}$ is a small constant which will be chosen after (8.1). Let $\mathcal{V}_0'$ be defined in the same way as $\mathcal{V}_0$ with β_1 replaced by $\beta(C_1)/4$. We will define $\tilde{Z}$ in $\mathcal{V}_0$ in the (q_1, q_2, Q_3) coordinates since by our choice of C_1, $\beta(C_1)$, and Proposition 3.1, they are alternate coordinates to (q_1, q_2, q_3) in $\mathcal{V}_2$.

Let $Z_{12}(q_1, q_2)$ be a pseudogradient vector field for $J_{12}(q_1, q_2)$ on $\tilde{\Lambda}_{12}$ or equivalently a pseudogradient vector field for J_{12} on Λ_{12}, which is invariant under translations in the sense of (2.24). We further require that Z_{12} generates a Morse-Smale flow under the level $M+1$. By this condition we mean the following: Consider

$$\frac{d\phi}{ds} = -Z_{12}(\phi), \quad \phi(0, (q_1, q_2)) = (q_1, q_2). \tag{4.17}$$

Note that any equilibrium point of this flow is a critical point of J_{12} and conversely. The flow is a Morse-Smale flow if the stable and unstable manifolds corresponding to any critical point of J_{12} intersect transversally in sections to the flow, e.g. on each noncritical level set. The existence of such a Z_{12} can be found e.g. as part of the proof of Theorem 7.2 for a finite dimensional case and in the proof of Theorem 8.2 for our case. Once Z_{12} has been obtained, we further require that all points (q_1, q_2) on the unstable manifolds of Z_{12} between the levels $\epsilon_1/2$ and $M+1$ satisfy (4.15). That this is possible follows from (4.8) and the surrounding paragraph related to the choice of C_1.

To get the pseudogradient flow in $\mathcal{V}_2$, we first define it in $\mathcal{V}_0$ where it is given by:

$$\begin{aligned}
&\frac{d}{ds}(q_1, q_2) = -Z_{12}(q_1, q_2)\\
&\frac{d}{ds}(Q_3 - [Q_3]) = -(Q_3 - [Q_3])\\
&\frac{d}{ds}\left[Q_3 - \frac{q_1 + q_2}{2}\right] = 0.
\end{aligned} \tag{4.18}$$

This will be denoted more succinctly in the (q_1, q_2, Q_3) coordinates by

$$\frac{d}{ds}(q_1, q_2, Q_3) = -\tilde{Z}_{12}(q_1, q_2, Q_3). \tag{4.19}$$

Let $\tilde{\omega}_{12}$ be a function such that

$$\tilde{\omega}_{12} \in C^\infty(\mathcal{V}_2, [0,1]) \tag{4.20}$$

$\tilde{\omega}_{12} = 1$ on $\mathcal{V}_0$, $\tilde{\omega}_{12} < 1$ on $\mathcal{V}_2 \backslash \mathcal{V}_0$, $\tilde{\omega}_{12} \geq \frac{1}{2}$ on $\mathcal{V}_0'$ and $\tilde{\omega}_{12} = 0$ on $\mathcal{V}_2 \backslash \mathcal{V}_1$. We extend $\tilde{Z}_{12}$ to $\mathcal{V}_1$ as follows:

$$
\begin{aligned}
&\frac{d}{ds}(q_1, q_2) = -Z_{12}(q_1, q_2) \\
(4.21) \qquad &\frac{d}{ds}(Q_3 - [Q_3]) = -(Q_3 - [Q_3]) \\
&\frac{d}{ds}\left[Q_3 - \frac{q_1+q_2}{2}\right] = (1 - \tilde{\omega}_{12}(q_1, q_2, Q_3))\frac{[Q_3 - \frac{q_1+q_2}{2}]}{\|[Q_3 - \frac{q_1+q_2}{2}]\|}.
\end{aligned}
$$

Next $\tilde{Z}_{12}$ is extended to $\mathcal{V}_2$. Let $-Y_{12}(q_1, q_2, q_3)$ be the vector field given by the right hand side of (4.21) expressed in the (q_1, q_2, q_3) coordinates. Observe that Y_{12} is defined on $\mathcal{V}_2$. Then our extension is via:

$$
(4.22) \qquad \frac{d}{ds}(q_1, q_2, q_3) = -\omega_{12}Y_{12} - (1 - \omega_{12})\tilde{I}'
$$

where ω_{12} is defined in (4.10), and $\omega_{12} = 1$ on $\mathcal{V}_1$ and $\omega_{12} = 0$ on $\Lambda\backslash\mathcal{V}_2$.

Carrying out this construction on each $\mathcal{V}_2(i,j)$, the resulting vector field, which we denote by $\tilde{Z}$, is globally defined and C^1. Consider the corresponding flow

$$
(4.23) \qquad \frac{dq}{ds} = -\tilde{Z}(q).
$$

The following lemma obtains for this flow. For convenience, it is stated for the case of $i = 1, \ j = 2$.

Lemma 4.24: Let

$$
\mathcal{V}_1'(1,2) = \{(q_1, q_2, Q_3) \mid \tilde{\omega}_{12}(q_1, q_2, Q_3) \geq \frac{1}{2}\}.
$$

Then there exist constants K and$\delta_1 > 0$ such that for any $q \in \mathcal{V}_2(1,2)\backslash\mathcal{V}_1'(1,2)$, we have:

$$
\tilde{I}'(q)\tilde{Z}(q) \geq \delta_1 \quad \text{and} \quad \|\tilde{Z}(q)\|_{W^{1,2}} \leq K
$$

provided that $\bar{\delta}$ is chosen small enough in (4.3).

Proof: The arguments in (4.3)-(4.8) show C_1 is independent of the approximation of I_{ij} by J_{ij}. $\mathcal{V}_2(1,2)$ is defined via (4.15)-(4.16) with C_1 replacing $C_1/4$. Since $\mathcal{V}_0(1,2) \cap (\mathcal{V}_2(1,2)\backslash\mathcal{V}_1'(1,2)) = \phi$, we have

$$
(4.25) \qquad \sum_{i=1}^{2} \|q_i - \frac{1}{2}[q_1 + q_2]\|_{L^\infty} > C_1/4
$$

or

$$\Psi(q) > \beta(C_1)/4 \tag{4.26}$$

for all $(q_1, q_2, Q_3) \in \mathcal{V}_2(1,2)\backslash\mathcal{V}_1'(1,2)$. Suppose (q_1, q_2) satisfies (4.25) but (4.26) does not hold for (q_1, q_2, Q_3). Since Z_{12} is a pseudogradient vector field for J_{12}, there is a $\gamma > 0$ such that

$$J_{12}'(q)Z_{12}(q) \geq \gamma\|J_{12}'(q)\|_{W^{1,2}}^2 \tag{4.27}$$

(where $q = (q_1, q_2)$). By our choice of $\tilde{C} = \tilde{C}\left(\frac{\epsilon_1}{2}, M+1\right)$, q must be outside of $N(\rho)$ which is a closed neighborhood of the critical set of J_{12}. Hence there exists an $\epsilon_0 > 0$ such that $\|J_{12}'(q)\|_{W^{1,2}}^2 \geq \epsilon_0$ and

$$J_{12}'(q)Z_{12}(q) \geq \epsilon_0\gamma > 0 \tag{4.28}$$

for all q outside of $N(\rho)$ and such that $J_{12}(q) \geq \epsilon_1 - \beta(C_1)/4 \geq \epsilon_1/2$ if $\beta(C_1) < \epsilon_1/2$. Thus for this case $J_{12} = I_{12}$ and

$$\tilde{I}'(q_1, q_2, q_3)Y_{12}(q_1, q_2, q_3) \geq J_{12}'(q_1, q_2)Z_{12}(q_1, q_2) \geq \epsilon_0\gamma \tag{4.29}$$

and

$$\begin{aligned}\tilde{I}'\tilde{Z}_{12} &= \omega_{12}\tilde{I}'Y_{12} + (1-\omega_{12})\|\tilde{I}'\|_{W^{1,2}}^2 \geq \\ &\geq \omega_{12}\epsilon_0\gamma + (1-\omega_{12})\|\tilde{I}'\|_{W^{1,2}}^2.\end{aligned} \tag{4.30}$$

Now by (4.25) and Corollary 3.48, there is a δ_0' such that $\|\tilde{I}'\|^2 \geq \delta_0'$ on $\mathcal{V}_2(1,2)\backslash\mathcal{V}_0'(1,2)$. Hence the lower bound for $\tilde{I}'\tilde{Z}$ follows in this case.

Next suppose (4.26) holds. Then, letting $\overline{Y}_{12}$ denote the 3^{rd} component of Y_{12}, we have

$$\begin{aligned}\Psi'(q)\overline{Y}_{12} = \int_0^1 |\dot{Q}_3|^2 dt + \frac{2}{(1+\|[Q_3 - \frac{1}{2}(q_1+q_2)]\|^2)^2}\cdot \\ \cdot (1-\tilde{\omega}_{12})\|[Q_3 - \frac{1}{2}(q_1+q_2)]\|\end{aligned} \tag{4.31}$$

Under (4.26), either

$$\int_0^1 |\dot{Q}_3|^2 dt \geq \beta(C_1)/8 \tag{4.32}$$

and then

$$\Psi'(q) \cdot \overline{Y}_{12} \geq \beta(C_1)/8$$

or

$$\frac{1}{1 + \|[Q_3 - \frac{1}{2}(q_1 + q_2)]\|^2} \geq \beta(C_1)/8 \tag{4.33}$$

in which case

$$\|[Q_3 - \frac{1}{2}(q_1 + q_2)]\|^2 \leq \frac{8}{\beta(C_1)} - 1. \tag{4.34}$$

Since $(q_1, q_2, Q_3) \in \mathcal{V}_2$, by (4.16)

$$\|[Q_3 - \frac{1}{2}(q_1 + q_2)]\|^2 \geq \frac{1}{\beta(C_1)} - 1. \tag{4.35}$$

Furthermore

$$\tilde{\omega}_{12}(q_1, q_2, Q_3) \leq \frac{1}{2}. \tag{4.36}$$

Hence by (4.31),

$$\Psi'\overline{Y}_{12} \geq \frac{\|[Q_3 - \frac{1}{2}(q_1 + q_2)]\|}{(1 + \|[Q_3 - \frac{1}{2}(q_1 + q_2)]\|^2)^2}$$

which by (4.33)-(4.35) is bounded from below by a positive number if $\beta(C_1)$ is small enough.

Thus in both cases (4.32) and (4.33),

$$\begin{aligned} \tilde{I}'\tilde{Z} &= \omega_{12}\tilde{I}'Y_{12} + (1 - \omega_{12})\|\tilde{I}'\|^2_{W^{1,2}} \\ &\geq \omega_{12}\Psi'\overline{Y}_{12} + (1 - \omega_{12})\|\tilde{I}'\|^2_{W^{1,2}} \\ &\quad - (1 - \omega_{12})\|\omega'_{12}\tilde{Z}\|_{W^{1,2}}|I_{12} - J_{12}| \\ &\quad - \omega_{12}\|I'_{12} - J'_{12}\|_{W^{1,2}}\|Z_{12}\|_{W^{1,2}}. \end{aligned} \tag{4.37}$$

Since $\|\tilde{I}'\|$ is bounded from below on $\mathcal{V}_2 \backslash \mathcal{V}_0$ by a positive constant by Corollary 3.48 and since $\Psi'\overline{Y}_{12}$ is similarly bounded from below, we have

$$\tilde{I}'\tilde{Z} \geq \delta_1 > 0 \tag{4.38}$$

provided that $|I_{12} - J_{12}|$ is small enough and as is proved below, $\|\tilde{Z}\|_{W^{1,2}}$ and $\|Z_{12}\|_{W^{1,2}}$ are uniformly bounded. Thus the first part of Lemma 4.24 is proved.

For the second part, observe that there is a constant K_1 such that $\|I'(q)\|_{W^{1,2}} \le K_1$ whenever

$$(4.39) \qquad I(q) \le M + 1.$$

Indeed

$$(4.40) \qquad \|I'(q)\|_{W^{1,2}} \le \|\dot{q}\|_{L^2} + \|V'(q)\|_{L^2}.$$

Now (4.34), (V_2), and (4) imply

$$(4.41) \qquad \|\dot{q}\|_{L^2} \le (2(M+1))^{1/2}.$$

By Proposition 2.1, $\|q_i - q_j\|_{L^\infty} \ge \delta$ where δ depends on $M+1$, and by $(V_1), (V_3)$, $V_{ij}(s) \le K_2(\delta)$ for $s \ge \delta$. Hence the existence of K_1 follows. Similarly there is a K_1' such that $\|I_{ij}'(q_i, q_j)\|_{W^{1,2}} \le K_1'$ whenever $I_{ij}(q_i, q_j) \le M+1$. Since $\|\tilde{I} - I\|_{C^2}$ and $\|J_{ij} - I_{ij}\|_{C^2}$ are small, we then get bounds for $\|\tilde{I}'\|_{W^{1,2}}$, $\|J_{ij}'\|_{W^{1,2}}$ similar to those for I', I_{ij}'. Hence we get bounds for Z_{ij}. The construction of $\tilde{Z}$ from Z_{ij} and t. bounds already obtained then yield the bound for $\tilde{Z}$.

Lemma 4.24 *has the following interesting consequence:*

Corollary 4.42: Let $q(s)$ be a trajectory of (4.23) with $\epsilon_1 \le \tilde{I}(q(0)) \le M+1$ and $s \ge 0$. Then there exists an $s_0 \ge 0$, depending on $q(0)$, i and j, such that for $s \ge s_0$, $q(s)$ either remains in $\mathcal{V}_1(i,j)$ or in $\Lambda \backslash \mathcal{V}_2(i,j)$.

Proof: Let $U \subset W$ be neighborhoods of $\mathcal{V}_1'$ such that $W \subset \mathcal{V}_2$ and $\operatorname{dist}(\partial U, \partial W) > 0$. Suppose $q(s)$ is a trajectory of (4.23) such that $q(s) \in W \backslash U$ for $s \in (s_1, s_2)$. Then for $s \in (s_1, s_2)$, the estimates of Lemma 4.24 apply and

$$(4.43) \qquad \begin{aligned} \tilde{I}(q(s_1)) - \tilde{I}(q(s_2)) &= -\int_{s_1}^{s_2} \tilde{I}'(q(s))\tilde{Z}(q(s))ds \\ &\le \delta_1(s_1 - s_2)) \end{aligned}$$

and

$$(4.44) \qquad \|q(s_2) - q(s_1)\|_{W^{1,2}} \le \int_{s_1}^{s_2} \|\tilde{Z}(q(s))\|_{W^{1,2}} \le K(s_2 - s_1).$$

Estimates (4.43) - (4.44) show that if $q(s_1) \in \partial U$ and $q(s_2) \in \partial W$, the change in $\tilde{I}$ produced by going from ∂U to ∂W can be estimated by

$$(4.45) \qquad \tilde{I}(q(s_1)) - \tilde{I}(q(s_2)) \le -\delta_1(s_2 - s_1) \le -\frac{\delta_1}{K} \quad \operatorname{dist}(\partial U, \partial W).$$

Now if $q(s)$ does not remain in $\mathcal{V}_1$ or in $\Lambda\backslash\mathcal{V}_2$ for all large s, either: (i) $q(s) \in \mathcal{V}_2\backslash\mathcal{V}_1$ for all large s, or (ii) $q(s)$ oscillates infinitely often between (a) $\partial\mathcal{V}_1$ and $\partial\mathcal{V}_2$, or (b) $\mathcal{V}_1$ and $\mathcal{V}_2\backslash\mathcal{V}_1$ or (c) $\Lambda\backslash\mathcal{V}_2$ and $\mathcal{V}_2$. If (i) occurred, we could apply (4.43) with $U = \mathcal{V}_1$, $W = \mathcal{V}_2$, and s_2 arbitrarily large. But this contradicts the fact that $\tilde{I} \geq 0$. If (ii) (a) occurred, the estimates (4.45) can be applied infinitely many times again contradicting that $\tilde{I} \geq 0$. The argument of case (i) in fact shows $q(s) \notin \mathcal{V}_2\backslash\mathcal{V}_1'$ for all large s and thus if (ii) (b) occurred, $q(s)$ must oscillate infinitely often between $\partial\mathcal{V}_1'$ and $\partial\mathcal{V}_1$. The argument of (ii) (a) excludes this possibility. Finally estimates of Lemma 4.24 can be shown to hold for a neighborhood of $\hat{\mathcal{V}}$ of $\mathcal{V}_2$ with $\text{dist}(\partial\hat{\mathcal{V}}, \partial\mathcal{V}_2) > 0$. Thus (ii) (c) follows from above arguments.

§5 Proof of Theorem 1.

In this section we show that $(V_1) - (V_6)$ imply that the set of critical values of I is unbounded. The proof relies in part on some technical results whose verification will be carried out in §8.

Since the proof is rather lengthy, we begin with a sketch. Suppose the set of critical values of I is bounded by a. Let $M > a$; a precise choice of M will be made later. Recall that for $s \in \mathbf{R}$, $I^s = \{y \in \Lambda \mid I(q) \leq s\}$. Let $\tilde{I}$ be given by (4.11) and let $\tilde{Z}$ be the pseudogradient vector field for $\tilde{I}$ constructed in §4. Finally let ϵ_1 be as defined in Proposition 2.9 and 2.9′. It will be shown in §8 that any trajectory $q(s)$ of (4.23) with $q(0) \in \tilde{I}^{M+1} = I^{M+1}$ which does not enter $\tilde{I}^{\epsilon_1}$ and I^{ϵ_1} or does not converge to a critical point of $\tilde{I}$ has a limit. The set of such limits, $\mathcal{H}$, will be called the set of *critical points at* ∞ of $\tilde{I}$ and will be characterized as

$$\begin{aligned}\mathcal{H} =&\{(\overline{q}_i,\overline{q}_j,Q_r) \in \Lambda_{ij} \times \mathbf{R}^{\ell} \mid (\overline{q}_i,\overline{q}_j) \text{ is a critical}\\ &\text{point for } J_{ij}, \epsilon_1 \leq J_{ij}(\overline{q}_i,\overline{q}_j) \leq M+1,\\ &\text{and } |Q_r - \frac{1}{2}[q_i+q_j]| \geq \frac{1}{\beta_1} - 1\}.\end{aligned}$$

An "unstable manifold", $W_u^\infty(\overline{q}_i,\overline{q}_j)$ will be associated with each such $(\overline{q}_i,\overline{q}_j)$. Namely $W_u^\infty(\overline{q}_i,\overline{q}_j)$ is the set of solutions of (4.23) whose limit set as $s \to -\infty$ has a nonempty intersection with $\mathcal{H}$. For a critical point q of $\tilde{I}$ in $\tilde{I}^{M+1}$, let $W_u(q)$ denote its unstable manifold for (4.23). Let $\mathcal{K}(J_{ij})$ denote the set of critical points for J_{ij} and $\mathcal{K}_{ij}^{M+1} = \mathcal{K}(J_{ij}) \cap J_{ij}^{M+1}$. Let $\mathcal{K}^{M+1}$ be the analogous set for $\tilde{I}$. Set

$$\mathcal{D}_{M+1} = \bigcup_{q \in \mathcal{K}^{M+1}} W_u(q)$$

and

$$\mathcal{D}_{M+1}^\infty = \bigcup_{i \neq j = 1}^{3} \quad \bigcup_{(\overline{q}_i,\overline{q}_j) \in \mathcal{K}_{ij}^{M+1}} W_u^\infty(\overline{q}_i,\overline{q}_j)$$

and let $\mathcal{W}^\infty \subset \mathcal{V}_2$ be a set with a piecewise smooth boundary which contains $I^{\epsilon_1} \cup \mathcal{D}_{M+1}^\infty$ in its interior. Let $\mathcal{V}_\epsilon = \mathcal{V}_\epsilon(\mathcal{D}_{M+1})$ be an ϵ neighborhood of $\mathcal{D}_{M+1}$. By 6° of Theorem 8.2, $\mathcal{W}^\infty$ may be chosen so that $\tilde{I}^{M+1} = I^{M+1}$ retracts by deformation onto

$$\mathcal{W}^\infty \cup \mathcal{V}_\epsilon \tag{5.1}$$

and $\mathcal{W}^\infty$ retracts by deformation onto $I^{\epsilon_1} \cup \mathcal{D}_{M+1}^\infty$. The sets $\mathcal{V}_\epsilon$ and $\mathcal{V}_\epsilon \cap \mathcal{W}^\infty$ are absolute *neighborhood retracts*, i.e. ANR's — see e.g. [10] — and their homologies vanish in

dimension $\geq m+1$ where m will be defined shortly. Roughly speaking, the Betti numbers (in rational homology) of (5.1) are uniformly bounded independent of M. (All references below to Betti numbers are in rational homology.) Hence by 6° of Theorem 8.2, the Betti numbers of Λ must be uniformly bounded. On the other hand, we will show that Λ can be characterized as the loop space of the set of pairwise distinct 3-tuples. By a theorem of Vigué-Poirrier-Sullivan [11], the Betti numbers of Λ are therefore unbounded. This contradiction establishes Theorem 1.

Carrying out the details of this sketch is a lengthy process. First we need some estimates for the (generalized) Morse indices of the critical points of $\tilde{I}$ in I^{M+1}. The critical points of I lie in I^a so by Proposition 2.1, there is a $\delta = \delta(a)$ such that for any critical point q of I in I^a, $(2.1)'$ holds. Moreover by Corollary 3.44, these critical points of I are uniformly bounded in E (up to a translation) by $C(\epsilon_1, a)$. For any such q and any $\varphi \in E$,

$$
\begin{aligned}
I''(q)(\varphi,\varphi) &= \int_0^1 \left(|\dot{\varphi}|^2 - \sum_{i\neq j=1}^{3} V_{ij}''(q_i - q_j)(\varphi_i - \varphi_j)(\varphi_i - \varphi_j) \right) dt \\
&= \int_0^1 \left(|\dot{\varphi}|^2 - \sum_{i\neq j=1}^{3} \sum_{k\neq n=1}^{3} \frac{\partial^2 V_{ij}}{\partial \xi_k \partial \xi_n}(q_i - q_j)(\varphi_{ik} - \varphi_{jk})(\varphi_{in} - \varphi_{jn}) \right) dt.
\end{aligned}
$$

The generalized Morse index of q is the dimension of the subspace of E on which $I''(q)$ is non-positive definite. The form of I'' and above remarks on δ and $C(\epsilon_1, a)$ show the Morse index of any critical point of I in I^a is bounded above by some $m = n(a) \in \mathbf{N}$. By (4.13) and Proposition 4.2, $\|I - \tilde{I}\|_{C^2}$ can be made as small as desired and critical points of $\tilde{I}$ lie in a small neighborhood of those of I. Hence m, being an integer, is also an upper bound for the generalized Morse index of any critical point of $\tilde{I}$. It can be shown that $\mathcal{D}_{M+1}$ is a Euclidean neighborhood retract — ENR — of dimension at most m. In any case, in the sequel m is the dimension of $\mathcal{D}_{M+1}$.

Next choose $k \in \mathbf{N}$ such that

$$
k \geq \max(m+1, 9\ell + 3). \tag{5.2}
$$

(A further restriction on k will be imposed later.) Let $\{z\} \in H_k(\Lambda, Q)$. Then $\{z\}$ may be represented by a chain z having support in a compact set $K \subset \Lambda$. Choose $M > a$ such that $K \subset I^{M+1}$. Then $\{z\}$ can be interpreted as a homology class in $H_k(I^{M+1}, Q)$. Let

$$
\mathcal{C} \equiv I^{\epsilon_1} \cup \mathcal{D}_{M+1}^{\infty}.
$$

Observe that $\mathcal{D}_{M+1} = \mathcal{D}_a$. For notational convenience we will generally drop the subscript $M+1$ from $\mathcal{D}$, $\mathcal{D}^\infty$ in what follows. $\mathcal{W}^\infty$ in (5.1) retracts by deformation on $\mathcal{C}$. Therefore

$$H_r(\mathcal{W}^\infty) \equiv H_r(C). \tag{5.3}$$

Since $\mathcal{V}_\epsilon$, $\mathcal{V}_\epsilon \cap \mathcal{W}^\infty$ and $\mathcal{W}^\infty$ are ANR's, the triad $(\mathcal{W}^\infty \cup \mathcal{V}_\epsilon, \mathcal{V}_\epsilon, \mathcal{W}^\infty)$ is excisive and the Mayer-Vietoris sequence holds:

$$\begin{aligned} \cdots \to H_{r+1}(\mathcal{W}^\infty \cup \mathcal{V}_\epsilon) \to H_r(\mathcal{W}^\infty \cup \mathcal{V}_\epsilon) \to \\ \to H_r(\mathcal{W}^\infty) \oplus H_r(\mathcal{V}_\epsilon) \to H_r(\mathcal{W}^\infty \cup \mathcal{V}_\epsilon) \to \cdots. \end{aligned} \tag{5.4}$$

(Here $H_r(A) \equiv H_r(A, \mathbf{Q})$). If $r > m$, $H_r(\mathcal{V}_\epsilon) = 0 = H_r(\mathcal{W}^\infty \cap \mathcal{D})$. Hence $H_r(\mathcal{W}^\infty \cup \mathcal{V}_\epsilon) = H_r(\mathcal{W}^\infty) = H_r(\mathcal{C})$ for $r > m$. Since $k > m$ and $\{z\}$ is a homology class of order k, $\{z\} \in H_k(\mathcal{C})$. Ideally we would like to interpret $\{z\}$ as an element of $H_k(\mathcal{D})$, i.e. drop I^{ϵ_1} from $\mathcal{C}$. This is not quite possible but something close to it is and will suffice for our purposes.

For $i \neq j \in \{1,2,3\}$ and $r \neq i, j$, note that by (4.15), there is a $C_1 > 0$ such that

$$\|q_r - \frac{1}{2}[q_i + q_j]\|_{L^\infty} + \|q_j - \frac{1}{2}[q_i + q_j]\|_{L^\infty} \le C_1/4 \tag{5.5}$$

whenever $(q_i, q_j, Q_r) \in \mathcal{D}^\infty$ and $J_{ij}(q_i, q_j) \ge \epsilon_1/2$. Let $\alpha(q_i, q_j)$ be chosen via Corollary 3.41 and Remark 3.43 (with $C(q_i, q_j)$ constrained by (5.5)) and further satisfying

$$\alpha(q_i, q_j) < \min\left(\frac{\epsilon_1}{2}, \beta(C_1)\right). \tag{5.6}$$

Define

$$W_{ij}^{\epsilon_1} = \{(q_i, q_j, Q_r) \in \Lambda_{ij} \times \mathbf{R}^\ell \mid J_{ij}(q_i, q_j) + \frac{1}{1 + |Q_r - \frac{1}{2}[q_i + q_j]|^2} = \epsilon_1$$

$$\text{and } \frac{1}{1 + |Q_r - \frac{1}{2}[q_i + q_j]|^2} \le \frac{\alpha(q_i, q_j)}{4}\}.$$

Note that since $J_{ij}(q_i, q_j) < \epsilon_1$ on $W_{ij}^{\epsilon_1}$, $I_{ij} = J_{ij}$ on this set and Corollary 3.41 and Remark 3.43 provide us with a diffeomorphism between (q_i, q_j, Q_r) and (q_i, q_j, q_r) coordinates provided that

$$\int_0^1 \frac{1}{2}|\dot{Q}_r|^2 dt + \frac{1}{1 + |Q_r - \frac{1}{2}[q_i + q_j]|^2} \le \frac{\alpha(q_i, q_j)}{4}. \tag{5.7}$$

But (5.7) is satisfied here since Q_r is a constant.

Let

$$Q_{ij} = \left\{ (q_i, q_j) \in \Lambda_{ij} \mid \epsilon_1 - \frac{\alpha(q_i, q_j)}{4} \le J_{ij}(q_i, q_j) < \epsilon_1 \right\}.$$

Note that $W_{ij}^{\epsilon_1}$ is a trivializable sphere bundle with fiber

$$F(q_i, q_j) = \left\{ Q_r \in \mathbf{R}^\ell \mid \frac{1}{1 + |Q_r - \frac{1}{2}[q_i + q_j]|^2} = \epsilon_1 - J_{ij}(q_i, q_j) \right\}$$

over Q_{ij}. By Proposition 2.2′, $J'_{ij} \neq 0$ in $J_{ij}^{\epsilon_1} \backslash J_{ij}^{\epsilon_1/2}$. Recalling (5.6) and further requiring

(5.8)
$$|\alpha'| \le \frac{1}{2} \inf_{w \in J_{ij}^{\epsilon_1} \backslash J_{ij}^{\epsilon_1/2}} \|J'_{ij}(w)\|,$$

a simple retraction argument and Proposition 2.9′ show Q_{ij} has the homotopy type of a subset of $(\mathbf{R}^\ell)^2$. Hence

(5.9)
$$H_r(W_{ij}^{\epsilon_1}, \mathbf{Q}) = 0$$

for $r \ge 3\ell - 1$.

Next set

(5.10)
$$Z_{ij} = \bigcup_{(\overline{q}_i, \overline{q}_j) \in \mathcal{K}_{ij}^{M+1}} \{ (q_i, q_j, Q_r) \in \Lambda_{ij} \times \mathbf{R}^\ell \mid (q_i, q_j) \in W_u(\overline{q}_i, \overline{q}_j) \text{ and}$$
$$\frac{1}{1 + |Q_r - \frac{1}{2}[q_i + q_j]|^2} \le \frac{\alpha(q_i, q_j)}{4} \}$$

and

(5.11)
$$\mathcal{C}_1 = I^{\epsilon_1} \cup \left(\bigcup_{i \neq j} Z_{ij} \right).$$

Since $\alpha(q_i, q_j) \le \beta(C_1)$, $Z_{ij} \subset \mathcal{D}^\infty$ and $\mathcal{C}_1 \subset \mathcal{C}$. Furthermore the choice of $\beta(C_1)$ implies the sets $Z_{ij} \backslash \text{int}\, I^{\epsilon_1}$ are pairwise disjoint. Working with the coordinates given by Corollary 3.41, it is not difficult to see that the injection of $\mathcal{C}_1$ in $\mathcal{C}$ is a homotopy equivalence. Therefore $\{z\}$ can be considered to be a homology class in C_1.

Set

$$\mathcal{B}_{ij} = Z_{ij} \backslash \text{int}\, I^{\epsilon_1}.$$

Note that

(5.12)
$$\mathcal{C}_1 = \bigcup_{i \neq j} (B_{ij} \cup I^{\epsilon_1})$$

and

(5.13) $$\mathcal{B}_{ij} \cap I^{\epsilon_1} = W_{ij}^{\epsilon_1}.$$

Moreover

(5.14) $$(\mathcal{B}_{12} \cup I^{\epsilon_1}) \cap (\mathcal{B}_{13} \cup I^{\epsilon_1} \cup \mathcal{B}_{32}) = I^{\epsilon_1}.$$

Let $\mathcal{C}_2 = \mathcal{B}_{13} \cup I^{\epsilon_1} \cup \mathcal{B}_{32}$. It is easy to check that the triad $(\mathcal{C}_1, \mathcal{B}_{12} \cup I^{\epsilon_1}, \mathcal{C}_2)$ is excisive. Thus the Mayer-Vietoris sequence applies and yields:

(5.15) $$H_{r+1}(\mathcal{C}_1) \to H_r(I^{\epsilon_1}) \to H_r(\mathcal{B}_{12} \cup I^{\epsilon_1}) \oplus H_r(\mathcal{C}_2) \to H_r(\mathcal{C}_1) \to \cdots.$$

By Proposition 2.9, I^{ϵ_1} has the homotopy type of a subset of $\mathbf{R}^{3\ell}$. Hence

(5.16) $$H_r(I^{\epsilon_1}) = 0$$

for $r \geq 3\ell + 1$,

(5.17) $$H_r(\mathcal{C}_1) = H_r(\mathcal{B}_{12} \cup I^{\epsilon_1}) \oplus H_r(\mathcal{C}_2).$$

A similar computation with $\mathcal{C}_2$ replacing $\mathcal{C}_1$ shows for $r \geq 3\ell + 1$,

(5.18) $$H_r(\mathcal{C}_2) = H_r(\mathcal{B}_{13} \cup I^{\epsilon_1}) \oplus H_r(\mathcal{B}_{32} \cup I^{\epsilon_1}).$$

Now we will study the homology of $\mathcal{B}_{12} \cup I^{\epsilon_1}$. We claim that the triad $(\mathcal{B}_{12} \cup I^{\epsilon_1}, I^{\epsilon_1}, \mathcal{B}_{12})$ is excisive. This will be shown in Lemma 5.23 below. Assuming it for now and recalling (5.13), by the Mayer-Vietoris sequence again,

(5.19) $$H_{r+1}(\mathcal{B}_{12} \cup I^{\epsilon_1}) \to H_r(W_{12}^{\epsilon_1}) \to H_r(\mathcal{B}_{12}) \oplus H_r(\mathcal{B}_{12} \cup I^{\epsilon_1}) \to H_r(\mathcal{B}_{12} \cup I^{\epsilon_1}) \to \cdots.$$

Now (5.9), (5.16), and (5.19) show

$$H_r(\mathcal{B}_{12}) = H_r(\mathcal{B}_{12} \cup I^{\epsilon_1})$$

for $r \geq 3\ell + 1$. Therefore (5.17) gives

$$H_r(\mathcal{C}_1) = H_r(\mathcal{B}_{12}) \oplus H_r(\mathcal{B}_{23}) \oplus H_r(\mathcal{B}_{31})$$

for $r \geq 3\ell + 1$. Thus $\{z\}$ can be expressed as a linear combination of closed chains having support in $\mathcal{B}_{12}, \mathcal{B}_{23}$, and $\mathcal{B}_{31}$.

Using Corollary 3.41, let C_{ij} be defined in the (q_i, q_j, Q_r) coordinates by

$$\frac{1}{2}\int_0^1 |\dot{Q}_r|^2 dt + \frac{1}{1 + ||[Q_r - \frac{1}{2}(q_i + q_j)]||^2} \le \alpha(q_i, q_j)$$

Hence $B_{ij} \subset C_{ij}$ so $\{z\}$ can be written as a linear combination of closed chains with support in the $C_{ij}$and $\{z\}$ lies in the subgroup of $H_k(\Lambda; \mathbf{Q})$ generated by the images of the $H_k(C_{ij}; \mathbf{Q})$ in $H_k(\Lambda; \mathbf{Q})$. Next let

$$D_{ij} = \{(q_i, q_j, Q_r) \in C_{ij} \mid Q_r = [Q_r]\}.$$

The map

$$[0,1] \times C_{ij} \to C_{ij}$$
$$(\theta, q_i, q_j, Q_r) \to (q_i, q_j, [Q_r] + (1-\theta)(Q_r - [Q_r]))$$

is a deformation retraction of C_{ij} onto D_{ij} is a bundle (which can be trivialized) over Λ_{ij} with fiber equal to the exterior of a ball. Hence the homology of D_{ij} is obtained by taking the tensor product of the homology of Λ_{ij} with $H_0(S^{\ell-1}) \oplus H_{\ell-1}(S^{\ell-1})$. Consequently for $k \ge \max(9\ell + 3, m)$,

$$\text{(5.20)} \qquad \operatorname{rank} H_k(\Lambda) \le \sum_{i \ne j = 1}^{3} [\operatorname{rank} H_{k-(\ell-1)}(\Lambda_{ij}) + \operatorname{rank} H_k(\Lambda_{ij})],$$

i.e. the k^{th} Betti number of Λ is bounded by a linear combination of the k^{th} and $(k-\ell+1)^{th}$ Betti numbers of Λ_{ij}. Since Λ_{ij} has the homotopy type of the free loop space on $S^{\ell-1}$, the Betti numbers, rank $H_r(\Lambda_{ij})$, are bounded independently of r [11]. Hence by (5.20), the Betti numbers of Λ are uniformly bounded. Note that this bound is independent of M.

We will show next that (5.20) does not hold for appropriately chosen k. Let $Y_j \subset (\mathbf{R}^{\ell})^j$ be the set of pairwise distinct j-tuples, $j = 2, 3$. Then Y_3 fibers over Y_2:

$$p : Y_3 \to Y_2$$
$$p(q_1, q_2, q_3) = (q_1, q_2)$$

where the fiber of p has the homotopy type of a wedge product of two spheres $S^{\ell-1}$. Since $\ell \ge 3$, Y_2 and hence Y_3 is simply connected. The cohomology ring of Y_3 needs at least two generators. Our space Λ is simply the set of $W^{1,2}$ loops in Y_3 and this is contained in the set of continuous loops in Y_3, the inclusion being a homotopy equivalence. Hence by a theorem of Vigué-Poirrier and Sullivan [11], the Betti numbers of Λ are unbounded. Now set

$$\text{(5.21)} \qquad \omega = \max_r \sum_{\substack{i \ne j = 1 \\ i<j}}^{3} [\operatorname{rank} H_{r-\ell+1}(\Lambda_{ij}) + \operatorname{rank} H_r(\Lambda_{ij})].$$

We further require that k satisfies

$$\operatorname{rank} H_k(\Lambda) \geq 1 + \omega. \tag{5.22}$$

This contradicts (5.20).

The following lemma now completes the proof of Theorem 1.

Lemma 5.23. The triple $(\mathcal{B}_{12} \cup I^{\epsilon_1}, I^{\epsilon_1}, \mathcal{B}_{12})$ is excisive.

Proof. Note first that I^{ϵ_1} has an open neighborhood in $\mathcal{B}_{12} \cup I^{\epsilon_1}$ which retracts on I^{ϵ_1}, namely int $I^{2\epsilon_1} \cap (\mathcal{B}_{12} \cup I^{\epsilon_1})$. This can be seen using the negative gradient flow for $\tilde{Z}$. (Indeed this fact can be used in the proof that $(C_1, \mathcal{B}_{12} \cup I^{\epsilon_1}, C_2)$ is excisive.)

To complete the proof, we need only show $\mathcal{B}_{12}$ has an open neighborhood in $\mathcal{B}_{12} \cup I^{\epsilon_1}$ which retracts on $\mathcal{B}_{12}$. It suffices to show that $W_{12}^{\epsilon_1}$ has an open neighborhood $\mathcal{O}$ in Λ which retracts on $W_{12}^{\epsilon_1}$ for then

$$(\mathcal{O} \cup \mathcal{B}_{12}) \cap (\mathcal{B}_{12} \cup I^{\epsilon_1}) = (\mathcal{O} \cap I^{\epsilon_1}) \cup \mathcal{B}_{12}$$

is an open neighborhood of $\mathcal{B}_{12}$ in $\mathcal{B}_{12} \cup I^{\epsilon_1}$ which retracts on $\mathcal{B}_{12}$. As was noted after (5.7), $W_{12}^{\epsilon_1}$ is a sphree bundle over Q_{12} with fiber at each point given by $F(q_1, q_2)$. To define an open set in Λ which retracts on $W_{12}^{\epsilon_1}$, we take a "larger" bundle over

$$Q_{12}^* = \{(q_1, q_2) \in \Lambda_{12} \mid \epsilon_1 - \frac{\alpha(q_1, q_2)}{2} < I_{12}(q_1, q_2) < \epsilon_1\}$$

with fiber at each point

$$\left\{ Q_3 \in \mathbf{R}^\ell \mid \frac{\epsilon_1 - I_{12}(q_1, q_2)}{2} < \frac{1}{2}\int_0^1 |\dot{Q}_3|^2 dt + \frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]|^2} \right.$$

$$\left. < 2(\epsilon_1 - I_{12}(q_1, q_2)) \right\}.$$

This latter set retracts continuously by deformation on the bundle over Q_{12}^* with fiber given by $F(q_1, q_2)$. Namely we contract $\dot{Q}_3$ to 0 and appropriately adjust $[Q_3]$ in the process. Then using the gradient flow for I_{12}', Q_{12}^* can be retracted by deformation onto Q_{12}. Since $W_{12}^{\epsilon_1}$ is a sphere bundle over Q_{12}, the retraction by defomration of the base space lifts to a retraction by deformation of the total space and the Lemma is proved.

Remark 5.24. In §6, the extension of Theorem 1 to the case where (V_6) does not hold will be studied. For that purpose, a sharper upper bound is needed for the smallest critical value of I. The following corollary to Theorem 1 provides us with such an estimate.

Corollary 5.25. Let ω be defined by (5.21). For $r \geq 3\ell + 1$ such that

$$\text{rank } H_r(\Lambda_j) \geq \omega + 1, \tag{5.26}$$

let $A \subset H_r(\Lambda)$ have rank at least $\omega + 1$. Let $K \subset \Lambda$ be compact and such that the support of one representative $z \subset K$ for all $\{z\} \in A$. Let $M \in \mathbf{R}$ be such that $K \subset I^{M+1}$. Then $\mathcal{K}^{M+1} \neq \phi$, i.e. I has a critical value in $I^{M+1} \backslash I^{\epsilon_1}$.

Proof. If not, the set N defined in §4 is empty and therefore by Proposition 4.2, $\mathcal{D}_{M+1} = \phi$. The number k in (5.2) can now be chosen independently of m and the argument involving (5.3) - (5.4) omitted. As earlier (5.20) holds for $k = r$. But this contradicts (5.21) and (5.26). Hence I has a critical value in I^{M+1}.

Remark 5.27. In the proof of Theorem 1, no explicit use was made of the fact that V is independent of t. Thus we also get:

Theorem 1′. Suppose $V = V(t,q) : \mathbf{R} \times F_3(\mathbf{R}^\ell) \to \mathbf{R}$ is T periodic in t and otherwise satisfies $(V_1) - (V_6)$. Then the functional

$$\int_0^T \left(\frac{1}{2}|\dot{q}|^2 - V(t,q) \right) dt \tag{5.28}$$

has an unbounded sequence of critical values which provide T periodic solutions of

$$\ddot{q} + V_q(t,q) = 0. \tag{5.29}$$

§6 Weaker potentials.

Our goal in this section is to study the effect of dropping hypothesis (V_6) in Theorem 1. To begin, recall that (V_6) implies Proposition 2.1 which forces any $q \in W^{1,2}$ for which $I(q) < \infty$ to be in Λ. If (V_6) is dropped, there are $W^{1,2}$ periodic functions which correspond to "collisions", i.e. $q_i(\tau) = q_j(\tau)$ for some $i \neq j$ and $\tau \in [0, T]$. If this happens, (HS) is not defined. Thus a notion of solution is required for this situation. Modifying [2], we say $q = (q_1, q_2, q_3) \in C(\mathbf{R}, (\mathbf{R}^\ell)^3)$ is a generalized T-periodic solution of (HS) if (5) (i) - (iv) of §1 holds. Now we have:

Theorem 6.1. If V satisfies $(V_1)-(V_5)$, Then for each $T > 0$, (HS) possesses a generalized T periodic solution.

Proof. Again we can take $T = 1$. An approximation argument in the spirit of [2, 13] will be used. Let $\chi \in C^\infty(\mathbf{R}, \mathbf{R})$ such that the $\chi(s) = 1$ if $s \le \frac{1}{2}$, $\chi'(s) \le 0$, and $\chi(s) = 0$ if $s \ge 1$. For each $\delta > 0$, let $\chi_\delta(s) = \chi\left(\frac{s}{\delta}\right)$. For $i \neq j \in \{1, 2, 3\}$, let $V_{ij}^\delta(x) = V_{ij}(x) - \delta|x|^{-2}\chi\delta(|x|)$. Then V_{ij}^δ satisfies $(V_1) - (V_6)$, $V_{ij}^\delta(x) = V_{ij}(x)$ if $|x| \ge \delta$ and

$$-V_{ij}^\delta(x) \ge -V_{ij}(x). \tag{6.2}$$

Set

$$V_\delta(q) = \sum_{i \neq j = 1}^{3} V_{ij}^\delta(q_i - q_j)$$

and

$$I_\delta(q) = \int_0^1 \left(\frac{1}{2}|\dot{q}|^2 - V_\delta(q)\right) dt.$$

By (6.2),

$$I_\delta(q) \ge I(q) \tag{6.3}$$

for all $q \in \Lambda$. Since V_δ satisfies the hypothesis of Theorem 1, for each $\delta > 0$, I_δ possesses an unbounded sequence of critical values. Moreover, by Corollary 5.25, I_δ possesses a critical value in $I_\delta^{M+1} \backslash I_\delta^{\epsilon_1}$ where a priori M and ϵ_1 depend on δ. Suppose $I_\delta(q) \le \epsilon_1$. Then the properties of $\mathcal{V}$ and choice of ϵ_1 imply $\|q_i - [q_i]\|_{L^\infty}$ is small and $|[q_i - q_j]|$ is large for $i \neq j \in \{1, 2, 3\}$. Hence for ϵ_1 small, $I_\delta(q) = I(q)$ for $q \in I^{\epsilon_1}$, i.e. ϵ_1 can be chosen independently of δ. Corollary 5.25 shows the choice of M depends on the compact set $K \subset \Lambda$. Hence M can be chosen independently of δ so that $K \subset I_\delta^{M+1}$ for all $\delta \in (0, 1)$. Thus for each such δ, there is a $q^\delta \in \Lambda \cap (I_\delta^{M+1} \backslash I_\delta^{\epsilon_1})$ such that q^δ is a critical point of I_δ.

We will show that as $\delta \to 0$, a subsequence of (q^δ) converges to a generalized 1-periodic solution of (HS). To prove this, observe first that Proposition 3.1 and Corollary 3.44 do

not require (V_6). An examination of their proofs shows that they hold uniformly for e.g. $\delta \in [0,1]$ and the constant $C(\epsilon_1, M)$ of Corollary 3.44 is independent of $\delta \in [0,1]$. Thus for each $\delta \in (0,1]$, we have

$$\sum_{i=1}^{3} \|q_i^\delta - v_\delta\|_{W^{1,2}} \leq C(\epsilon_1, M) + 1 \tag{6.4}$$

where $v_\delta = \frac{1}{2}[q_1^\delta + q_2^\delta]$. Since $q^\delta - \psi(v_\delta)$ is also a critical point of I_δ corresponding to the same critical value, by (6.4) a subsequence of these critical points converge weakly in $W^{1,2}$ and strongly in L^∞ to $q \in W^{1,2}$. Moreover

$$-\int_0^1 V(q(t))dt \leq M + 1. \tag{6.5}$$

Indeed for all $\delta \in (0,1]$,

$$-\int_0^1 V_\delta(q^\delta(t))dt \leq M + 1. \tag{6.6}$$

Consequently for $\epsilon > 0$,

$$-\int_0^1 \sum_{i \neq j = 1}^{3} (1 - \chi_\epsilon(|q_i^\delta - q_j^\delta|))V_{ij}^\delta(q_i^\delta - q_j^\delta)dt \leq M + 1. \tag{6.7}$$

Letting $\delta \to 0$, it readily follows from (6.7) that

$$-\int_0^1 \sum_{i \neq j = 1}^{3} (1 - \chi_\epsilon(|q_i - q_j|))V_{ij}(q_i - q_j)dt \leq M + 1. \tag{6.8}$$

Thus letting $\epsilon \to 0$ in (6.8) yields (6.5). Hence q satisfies (iii) of (5). Next (6.8) and (V_4) imply that $\mathcal{D}$, as defined in (5) (i), has measure 0 and (5) (i) holds. If $\tau \in [0,1]\backslash\mathcal{D}$, there is an $\epsilon, \rho > 0$ such that if $|t - \tau| \leq \rho$, $|q_i(t) - q_j(t)| \geq \epsilon$ for each $i \neq j \in \{1,2,3\}$. The system of differential equations:

$$\ddot{q}^\delta + V_\delta'(q^\delta) = 0 \tag{6.9}$$

shows $q^\delta(t) \to q(t)$ in C^2 for $|t - \tau| \leq \rho$ and q satisfies (HS) on this set. Thus (5) (ii) holds. Lastly (5) (iv) is valid for q^δ and V_δ with $\mathcal{D}_\delta = \phi$ and a corresponding constant γ_δ. Hence on passing to a limit, we get (5) (iv). The proof of Theorem 6.1 is complete.

Corollary 6.10. If $\mathcal{V}$ satisfies $(V_1) - (V_5)$ and

$$V'(q) \neq 0 \quad \text{for all} \quad q \in (\mathbf{R}^\ell)^3, \tag{V_7}$$

then (HS) has infinitely many distinct generalized T-periodic solutions.

Proof. We use a standard argument. By Theorem 6.1, (HS) has a generalized T-periodic solution q_T. By (V_7), q_T is not an equilibrium solution and therefore its minimal period is T/k_1 for some $k_1 \in \mathbb{N}$. Invoking Theorem 6.1 again with T replaced by $T/2k_1$, we find a second nonequilibrium generalized T-periodic solution with minimal period $T/k_1 M T/k_1$. Repeating this process gives the result.

As with Theorem 5.28, the proof of Theorem 6.1 yields

Theorem 6.11. If $\mathcal{V} = \mathcal{V}(t,q)$ is T periodic in t and satisfies $(V_1)-(V_5)$, then (5.30) has at least one generalized T-periodic solution.

§7. The Retraction Theorem and Related Results: The Finite Dimensional Case.

A key fact used in the proof of Theorem 1 was that $\tilde{I}^{M+1}$ retracts by deformation onto the set given in (5.1). In this and the following section we will establish this fact together with some related results. This will be done in two stages. First in this section we will prove an analogue of Theorem 8.2 for a Morse function on a compact manifold. Then in §8, it will be indicated how to modify this simpler situation to get Theorem 8.2.

To begin, recall if $f \in C^1(\mathcal{O}, \mathbf{R})$ where $\mathcal{O} \subset \mathbf{R}^j$, Ψ is a pseudogradient vector field for f on $\mathcal{O}$ if Ψ is defined and locally Lipschitz continuous on $\{y \in \mathcal{O} \mid f'(y) \neq 0\}$ and there are constants $\beta < \alpha$ such that for all y in this set:

$$(7.1)\qquad \begin{cases} |\Psi(y)| \leq \alpha |f'(y)| \\ f'(y)\Psi(y) \geq \beta |f'(y)|^2. \end{cases}$$

Let $\mathcal{K}(f)$ denote the set of critical points of f. Our main result in this section is the following:

Theorem 7.2. Let $\mathcal{M}(\subset \mathbf{R}^m)$ be a compact manifold and $f \in C^2(\mathcal{M}, \mathbf{R})$ be a Morse function. Let Ψ be a C^1 pseudogradient vector field for f such that Ψ extends to all of $\mathcal{M}$ as a C^1 function and such that the critical points of f are nondegenerate zeros of Ψ. Then in any C^1 neighborhood of Ψ, there exists another pseudogradient vector field, Φ, of f satisfying:

1° The critical points of f are nondegenerate zeros of Φ;

2° If $x, y \in \mathcal{K}(f)$, the unstable manifold of x, $W_u(x)$ (for the flow generated by Φ) and the stable manifold of y, $W_s(y)$, intersect transversally;

3° If $x \in \mathcal{K}(f)$ and

$$F_x = \{y \in \mathcal{K}(f) \backslash \{x\} \mid W_u(x) \cap W_s(y) \neq \phi\},$$

then

$$\overline{W_u(x)} = W_u(x) \cup \Big(\bigcup_{y \in F_x} W_u(y) \Big);$$

4° If $z \in F_x$, there is an $r_0 > 0$ and a family of neighborhoods, U_r, $0 < r < r_0$, of $W_u(z)$ satisfying

(i) $U_r \subset U_t$ if $r < t$

(ii) U_r is a trivial bundle over $W_u(z)$ with fiber homeomorphic to $(W_u(x) \cap W_s(z)) \cup \{z\}$,

(iii) The trace of $W_u(x)$ in U_r is a trivial subbundle over $W_u(z)$ with fiber diffeomorphic to $W_u(x) \cap W_s(z)$,

(iv) $\bigcap_{r \leq r_0} U_r = W_u(z)$,

(v) The diameter of the fibers tends to 0 as $r \to 0$.

5° Let $f^c = \{x \in \mathcal{M} \mid f(x) \leq c\}$. If $a < b$ are noncritical values of f and

$$W_u(a,b) \equiv f^a \cup \{W_u(x)) \mid x \in \mathcal{K}(f), a < f(x) < b\},$$

then $W_u(a,b)$ is an ENR and f^b retracts by deformation onto $W_u(a,b)$.

Remark 7.3. Actually a stronger statement than 5° is proved in [15], namely that $W_u(a,b)$ admits an isolating block in the sense of Conley [14] which retracts by deformation on $W_u(a,b)$.

We will now carry out the proof of Theorem 7.2. Statement 1° follows from (7.1) and that Φ is C^1 close to Ψ. An induction argument will be used to prove 2° – 4° and for this some preliminaries are required. Suppose $\dim \mathcal{M} = \ell$. Let $y \in \mathcal{K}(f)$. Then there are local coordinates near y, given e.g. by the Morse Lemma, such that $\mathbf{R}^\ell$ splits into $E^- \oplus E^+$ where E^- is the unstable manifold of y for Φ and E^+ is the corresponding stable manifold. Moreover, if X is the coordinate along E^- and Y along E^+, near y

$$f(X,Y) = f(y) - |X|^2 + |Y|^2. \tag{7.4}$$

Suppose $x \in \mathcal{K}(f) \backslash \{y\}$ and $W_u(x)$ is transverse to E^+. We want to understand how $W_u(x)$ behaves along E^-. Since both $W_u(x)$ and E^+ are invariant under the flow generated by Φ, transversality here means, of course, transversality in sections to the flow lines; i.e.

transversality of the intersections of these two sets with

$$P_\rho \equiv \{(X,Y) \mid |X| \le \rho, |Y| = \rho\}$$

for ρ small enough. Thus we are assuming that for $0 < \rho \le \rho_0 < 1$, the two manifolds:

$$W_u(x) \cap P_\rho = \{(X,Y) \in W_u(x) \mid |X| \le \rho, |Y| = \rho\}, \tag{7.5}$$

and

$$S_\rho = \{(0,Y) \mid |Y| = \rho\} \tag{7.6}$$

intersect transversally in P_ρ. The intersection is then a manifold which we will denote by $\mathcal{T}_\rho$. In a neighborhood of a point of $\mathcal{T}_\rho$, $W_u(x) \cap P_\rho$ may be thought of as a vector bundle over $\mathcal{T}_\rho$ with fibers parallel to E^- since, by a version of the Implicit Function Theorem, in such a neighborhood, any point of $W_u(x) \cap P_\rho$ may be represented by an associated point on $\mathcal{T}_\rho$ and an abcissa on E^-. This representation has a local character in general, i.e. it cannot be extended to all of $W_u(x) \cap P_{\rho'}$, even for a $\rho' < \rho$, unless $\mathcal{T}_\rho$ is compact or some other special feature occurs.

We *assume* that there is a C^1 diffeomorphism ψ such that for a $\rho' < \rho$,

$$\psi(W_u(x) \cap P_{\rho'}) \cong \{X \in E^- \mid |X| \le \rho'\} \times \mathcal{T}_\rho \tag{7.7}$$

where $\cong$ denotes the diffeomorphism ψ. When (7.7) holds, we say $W_u(x) \cap W_s(y)$ intersect transversally *in a uniform way.*

In order to understand how $W_u(x)$ behaves along E^-, let γ be free for the moment and let $(X_0, 0)$, $|X_0| \le \gamma$ be a point of E^-. Let $\theta \in \mathbf{R}$ satisfy

$$0 < \theta < (\rho')^2 \tag{7.8}$$

and set

$$S_\theta(X_0) = \{(X_0, Y) \mid |Y| = \theta\}. \tag{7.9}$$

We will describe the set $S_\theta(X_0) \cap W_u(x)$. Locally the flow corresponding to Φ is given by

$$\eta(t, X, Y) = (e^{-t}X, e^tY). \tag{7.10}$$

This formula holds in a neighborhood of y, i.e. if

$$\begin{cases} |e^tY| \le \alpha; |X| \le \alpha < 1 & \text{for } t \ge 0 \\ |e^{-t}X| \le \alpha; |Y| \le \alpha < 1 & \text{for } t \le 0 \end{cases} \tag{7.11}$$

holds with a suitable α. We take $\gamma = \alpha$ and ρ such that

$$\rho \le \min(\rho_0, \alpha) < 1. \tag{7.12}$$

If $z = (X_0, Y) \in S_\theta(X_0) \cap W_u(x)$, then

$$\left(\frac{\theta}{\rho'}X_0, \frac{\rho'}{\theta}Y\right) \in W_u(x) \cap P_{\rho'}. \tag{7.13}$$

Indeed since $|X_0| \le \gamma = \alpha < 1$ by (7.8) - (7.9), $\theta(\rho')^{-1}|X_0| < \rho'$ and $\rho'\theta^{-1}|Y| = \rho'$. Thus $(\theta(\rho')^{-1}X_0, \rho'\theta^{-1}Y) \in P_{\rho'}$. Furthermore setting

$$e^{t_0} = \frac{\rho}{\theta} \ge \frac{1}{\rho'} \ge 1, \tag{7.14}$$

$(\theta(\rho')^{-1}X_0, \rho'\theta^{-1}Y) = (e^{-t_0}X_0, e^{t_0}Y)$ with $t_0 \ge 0$. Condition (7.11) is satisfied. Thus $(\theta(\rho')^{-1}X_0, \rho'\theta^{-1}Y) = \eta(t_0, z)$ and since $z \in W_u(x)$, $(\theta(\rho')^{-1}X_0, \rho'\theta^{-1}Y) \in W_u(x)$. Hence (7.13) holds.

Conversely let $\rho' \ge \lambda > 0$ and $z' = (\lambda X_0, Y) \in W_u(x) \cap P_{\rho'}$. Then $z = (X_0, \lambda Y) \in S_{\rho\lambda}(X_0)$. Setting

$$e^{-t_1} = \frac{1}{\lambda} \ge \frac{1}{\rho'} \ge 1, \tag{7.15}$$

we have $t_1 \le 0$ and

$$|e^{-t_1}\lambda X_0| = |X_0| \le \alpha; |Y| = \rho' < \alpha. \tag{7.16}$$

Therefore (7.10) holds on $[t_1, 0]$ and

$$z = \eta(t_1, z') \in W_u(x). \tag{7.17}$$

Choosing $\lambda = \theta(\rho')^{-1}$ in (7.7), (7.10), and (7.17) establish a diffeomorphism between $S_\theta(X_0) \cap W_u(x)$ and $\{\theta(\rho')^{-1}X_0\} \times \mathcal{T}_\rho$. This diffeomorphism is given by $\eta(t_0, \cdot) \circ \psi$ where t_0 satisfies (7.14). Alternatively with $t_0 = \log(\rho'\theta^{-1})$,

$$\eta(-t_0, \cdot) \circ \psi^{-1}\left(\left\{\frac{\theta}{\rho'}X_0\right\} \times \mathcal{T}_\rho\right) = S_\theta(X_0) \cap W_u(x). \tag{7.18}$$

Observe that the restriction $|X_0| \leq \gamma$ plays no qualitative role in what was done above. Once (7.18) is established for $|X_0| \leq \gamma$, it holds for any $X_0' \in E^-$ or more generally in $W_u(y)$. Indeed for any such point,

$$\lim_{t\to\infty} \eta(t, X_0') = y.$$

Thus we may choose $t(X_0') > 0$ such that $|X_0| = |\eta(t(X_0'), X_0')| \leq \gamma$ and (7.18) then holds at X_0. The result then transports to X_0' through $\eta(t(x_0'), \cdot)$. Moreover (7.18) extends in a natural way to

$$B_{\rho'}(X_0) \cap W_u(x) = (\{(X_0, Y) \mid |Y| \leq \rho'\} \cap W_u(x)) \cup \{(X_0, 0)\} \tag{7.19}$$

through the map

$$\begin{aligned} &[0, \rho'] \times \mathcal{T}_\rho/\{0\} \times \mathcal{T}_\rho \to B_{\rho'}(X_0) \cap W_u(x) \\ &\qquad (t, Y) \to \eta(-t_0, \psi^{-1}(tX_0, Y)), \qquad t_0 = -\log t \quad \text{if} \quad t \neq 0 \\ &\qquad (0, Y) \to (X_0, 0) \quad \text{if} \quad t = 0. \end{aligned} \tag{7.20}$$

Clearly (7.20) extends to all $X_0 \in W_u(y)$ using the same argument as for extending (7.18). Observe also that $[0, \rho'] \times \mathcal{T}_\rho/\{0\} \times \mathcal{T}_\rho$ is homeomorphic to $(W_u(x) \cap W_s(y)) \cup \{x\}$. Indeed

$$W_u(x) \cap W_s(y) \quad \text{is diffeomorphic to} \quad \mathcal{T}_\rho \times (0, \infty), \tag{7.21}$$

the diffeomorphism being given by the flow, i.e.

$$z \in W_u(X) \cap W_s(y) \to (\eta(t(z), z), t(z)) \in \mathcal{T}_\rho \times (0, \infty) \tag{7.22}$$

where $t(z)$ is the unique value of t such that $\eta(t(z), z) \in S_\rho$. This diffeomorphism can easily be modified to be a diffeomorphism to $\mathcal{T}_\rho \times (0, \rho')$. It maps a deleted neighborhood of x in $(W_u(x) \cap W_s(y)) \cup \{x\}$ into a neighobrhood of $\mathcal{T}_\rho \times \{0\}$ in $\mathcal{T}_\rho \times [0, \rho')$ and then naturally extends to a homeomorphism which is a diffeomorphism outside any given neighborhood of $\{x\}$:

$$(W_u(x) \cap W_s(y)) \cup \{x\} \longrightarrow [0, \rho') \times \mathcal{T}_\rho / \{0\} \times \mathcal{T}_\rho. \tag{7.23}$$

The above observations combine to give the following result:

Proposition 7.24. Assume $W_u(x)$ and $W_s(y)$ intersect transversally in a uniform way, i.e. (7.7) holds for suitable constants $\rho' < \rho \leq \rho_0$. Then $W_u(y)$ is contained in $\overline{W_u(x)}$ and there is a decreasing sequence U_r, $r \leq r_0$, of neighborhoods of $W_u(y)$ in $W_u(x) \cup W_u(y)$ which are trivial bundles over $W_u(y)$ — see (7.20) - (7.23) — with fiber homeomorphic to $(W_u(x) \cap W_s(y)) \cup \{y\}$. Moreover, $U_r \backslash W_u(y)$ is a subbundle over $W_u(y)$ with fiber diffeomorphic to $W_u(x) \cap W_s(y)$. The diameter of the fiber tends to 0 as $r \to 0$.

In order to continue the proof of Theorem 7.2, a stronger notion of transversality than that given by (7.7) is needed. Let

$$\psi : W_u(x) \cap P_{\rho'} \to \{X' \in E^- \mid |X'| \leq \rho'\} \times \mathcal{T}_\rho$$

so $\psi^{-1}(X', Y') = (X, Y)$. We assume hereafter that there is a uniform $\sigma > 0$ such that if P_- and P_+ denote the projectors on E^- and E^+, we have

$$\sup_{(X',Y') \in \psi(W_u(x) \cap P_{\rho'})} \; \sup_{e^- \in E^- \backslash \{0\}} \frac{\|P_+(D\psi^{-1}_{(X',Y')}((e^-, 0))\|}{\|P_-(D\psi^{-1}_{(X',Y')}((e^-, 0))\|} \leq \sigma \tag{7.25}$$

where $\|\cdot\|$ is a norm on the tangent space to $\mathcal{M}$.

Condition (7.25) means a uniform transversality in a strong sense since it relies on the fact that there is a transversality coefficient σ uniformly along $\psi^{-1}(W_u(x) \cap P_{\rho'})$. For a simple transversality, σ may depend on (X, Y).

We will say $W_u(x)$ and E^+ intersect transversally *strongly and uniformly* if (7.25) holds.

Now let $K \subset W_u(y)$ be compact. Let

$$g : U_r \to W_u(y) \tag{7.26}$$

be the fibration associated with Proposition 7.24. Since $W_u(y)$ is contractible, its tangent bundle is trivial as is the tangent bundle of $\mathcal{M}$ along $W_u(y)$. This yields an extension of the tangent bundle of $W_u(y)$ to a neighborhood of K in U_r, i.e. a total space $F(K)$ which is a vector bundle over $U_r(K)$, a neighborhood of K in U_r:

$$g_1 : F(K) \to U_r(K) \tag{7.27}$$

such that

$$g_1 \Big|_{g_1^{-1}(K)} : g_1^{-1}(K) \to K \tag{7.28}$$

is the tangent bundle to $W_u(y)$ restricted to K. Taking r small enough, we may assume that

$$U_r(K) = g^{-1}(K) \tag{7.29}$$

where g is defined in (7.26). The fibers of g_1 are of course diffeomorphic to the tangent space to $W_u(y)$ at a given point. Heuristically, by continuity, the direction of a g_1 fiber at $z \in g^{-1}(K)$ approaches that of the tangent space to $W_u(y)$ at $g(z)$ as $z \to g(z)$. Let μ be a metric on $\mathcal{M}$ and let

$$g_2 : F(K)^{\perp} \to U_r(K) = g^{-1}(K) \tag{7.30}$$

be the normal bundle to $F(K)$.

Proposition 7.31. There exists a subbundle of the tangent bundle to $W_u(x)$ along $g^{-1}(K) - K$:

$$g_3 : G(K) \to g^{-1}(K) - K$$

whose fiber is a subspace of the tangent space to $W_u(x)$ at the same point, of dimension equal to $\dim W_u(y)$. Moreover $G(K)$ has the following property: For any $z \in g^{-1}(K) - K$ and any vector $v_z \in (G(K))_z$, the fiber at z for g_3, splits naturally into $h_z + k_z$ where $h_z \in (F(K))_z$, the fiber at z of g_1, and $k_z \in (F(K)^{\perp})_z$, the fiber at z of g_2. Then

$$\lim_{z \to g(z)} \frac{\|k_z\|}{\|h_z\|} = 0 \tag{7.32}$$

uniformly in z and v_z or equivalently

$$\lim_{\tau \to 0} \sup_{z \in U_\tau(K) - K} \sup_{v_z \in (G(K))_z \setminus \{0\}} \frac{\|k_z\|}{\|h_z\|} = 0 \tag{7.33}$$

in the norm associated with the metric μ.

Remark 7.34. Conditions (7.32)-(7.33) together with the fact that g_1 extends the tangent bundle to $W_u(y)$ restricted to K ((7.27)-(7.28)) means that the tangent space to $W_u(x)$ contains a subbundle in a neighborhood of K which extends the tangent bundle to $W_u(y)$ along K.

Proof of Proposition 7.31. We use (7.25) and (7.20). The latter tells us that for any $z \in U_\tau(K) \setminus K$, $\tau \leq r$, there exists $t_0(z)$, $t \in (0, \rho')$, $t = e^{-t_0(z)}$, and $Y \in \mathcal{T}_\rho$ such that

$$z = \eta(-t_0(z), \psi^{-1}(t(z)g(z), Y)) \tag{7.35}$$

where $\lim_{\tau \to 0} t(z) = 0$, $\lim_{\tau \to 0} t_0(z) = \infty$. Equation (7.35) is written in the (X, Y) coordinates. Instead of (7.35), we will write:

$$z = \eta(-t_0(z), \psi^{-1}(X', Y')) \tag{7.36}$$

where $(X', Y') \in \{X' \in E^- \mid |X'| \leq \rho'\} \times \mathcal{T}_\rho$, and $(t(z)g(z), Y) = (X', Y')$. Since $\lim_{\tau \to 0} t(z) = 0$, we can use ψ if τ is small enough.

The flow $\eta(t, \cdot)$ expands coordinates in the direction of $W_u(y)$ by a factor of e^{-t} and contracts in the direction of $W_s(y)$ by a factor of e^t as $t \to -\infty$. Therefore using (7.25) and setting

$$G(K)_z = D\eta_{(-t_0(z), \psi^{-1}(X', Y'))} \circ D\psi^{-1}_{(X', Y')}(E^- \times \{0\}). \tag{7.37}$$

Proposition 7.31 follows.

Now we are ready to prove $2° - 4°$ of Theorem 7.2. This statement will be proved by induction on the critical values. Let $c_1 < \cdots < c_m$ be the critical values of f. For the sake of simplicity, each critical value will be assumed to correspond to a single critical point. The set of critical points is $\{x_1, \cdots, x_m\}$ and they are nondegenerate. Now $W_u(x_i)$ and $W_s(x_i)$ are the stable and unstable manifolds of x_i for Φ. We are going to perturb Φ in the course of the proof. This of course causes perturbations of $W_u(x_i)$ and $W_s(x_i)$. Nevertheless for convenience the same notation will be used for the perturbed manifolds.

Statements $2° - 4°$ hold for the minimum, x_1, since they are vacuous. Using the Morse Lemma, they also hold for x_2. In the induction below, c is a noncritical value. Let

$$f_c = \{x \in \mathcal{M} \mid f(x) \geq c\}$$

and $\mu(A, B)$ denote the distance between sets A and B. Let

$$2 \leq p \leq m \tag{7.38}$$

be given. Recall that

$$F_{x_i} = \{x_j \mid W_u(x_i) \cap W_s(x_j) \neq \phi\}.$$

We assume inductively that

(i) If $W_u(x_i) \cap W_s(x_j) = \phi$, $\mu(W_u(x_i), \ W_s(x_j)) \geq \rho_{ij} > 0$, $i, j \leq p-1$; if $W_u(x_i) \cap W_s(x_j) \neq \phi$, $W_u(x_i)$ and $W_s(x_j)$ intersect transversally for $i, j \leq p-1$ and the intersection is uniform and strong in the sense of (7.25).

(ii) $\overline{W_u(x_i)} \cap f_c = (W_u(x_i) \cup \{W_u(x_j) \cap f_c \mid x_j \in F_{x_i} \text{ and } c_j > c\}$ for all noncritical values $c < c_i$, $i \leq p-1$.

(iii) For i given, $i \leq p-1$, let $\mathcal{O}((\bigcup_{x_j \in F_{x_i}} W_u(x_j))$ be an open neighborhood of $\bigcup_{x_j \in F_{x_i}} W_u(x_j)$. Then for any such $\mathcal{O}$ and any noncritical value $c < c_i$, $(W_u(x_i) \backslash \mathcal{O}) \cap f_c$ is compact.

Conditions (i)-(iii) are obviously satisfied for $p - 1 = 2$. Later we are going to prove that they are satisfied for any p after a suitable perturbation of Φ. Assuming for now that

(i)-(iii) have been established, we will show that (i)-(iii) imply $2° - 4°$. Indeed (i) implies $2°$ and (ii) implies $3°$. Lastly $4°$ follows from Proposition 7.24 and 7.31 and the strong and uniform intersection property of (i).

Now we will prove (i)-(iii). Clearly (iii) follows from (ii). We will prove (i) and (ii) by induction. We assume (i) and (ii) hold for $i, j \leq p-1$. As was already observed, (i) and (ii) hold for $i, j \leq 2$. Three steps are needed to get the result for p.

Step 1. (i) and (ii) hold, after possibly perturbing Φ, for $i = p,\ j = p-1$, and $c_{p-1} < c < c_p$.

In order to verify (i), we perturb the compact manifolds $W_u(x_p) \cap f^{-1}(c)$ and $W_s(x_{p-1}) \cap f^{-1}(c)$, so that they intersect transversally. This corresponds, e.g. to perturbing Φ along the normal bundle to $W_u(x_p) \cap f^{-1}(c)$ in $f^{-1}(c)$. The resulting transversal intersection is then uniform and strong. Now (ii) is immediate since $\overline{W_u(x_p)} \cap f_c = W_u(x_p) \cap f_c$.

Step 2. If (i) is satisfied for $i = p,\ r \leq j \leq p-1$, and (ii) holds for $i = p,\ c_r < c < c_p$, with c a noncritical value, then (ii) holds for $i = p$ and noncritical $c,\ c_{r-1} < c < c_p$.

Here we consider two cases:

Case 1. $W_u(x_p) \cap W_s(x_r) = \phi$.

Then $\mu(W_u(x_p), W_s(x_r)) \geq \rho_{rp} > 0$ and the classical deformation theorems tell us that if $c_r < c' < c_{r+1}$ and $c_{r-1} < c < c_r$, then $W_u(x_p) \cap f_c = \eta(-1, W_u(x_p) \cap f_{c'})$, for a suitable renormalization of the flow for $-f'$ (so that $f_{c'} \backslash \mathcal{O}_{rp}$ is deformed into f_c in "time" -1 where $\mathcal{O}_{rp}$ is a uniform $\frac{1}{2}\rho_{rp}$ neighborhood of $W_s(x_r)$). Therefore $\overline{W_u(x_p) \cap f_c} = \eta(-1, \overline{W_u(x_p) \cap f_{c'}})$ since $\eta(-1, \cdot)$ is invertible and since (ii) holds for c',

$$\overline{W_u(x_p)} \cap f_c = W_u(x_p) \cup \{W_u(x_j) \cap f_c \mid x_j \in F_{x_p} \text{ and } c_j > c\}. \tag{7.39}$$

Hence Step 2 follows for this case.

Case 2. $W_u(x_p) \cap W_s(x_r) \neq \phi$.

Then by (i), these manifolds intersect transversally, strongly and uniformly. From the proofs of Propositions 7.24 and 7.31, $W_u(x_p) \cup W_u(x_r)$ fibers locally over $W_u(x_r)$ with a fiber homeomorphic to $(W_u(x_p) \cap W_s(x_r)) \cup \{x_r\}$ and $W_u(x_p)$ locally is a subbundle with a fiber homeomorphic to $W_u(x_p) \cap W_s(x_r)$ (contained in $W_u(x_p) \cup W_s(x_r) \cup \{x_r\}$). These fibrations are transversal to the flow of Φ (see (7.18)-(7.20)) which leaves $W_u(x_r)$ invariant. Therefore for $c_{r-1} < c < c_r$, $(W_u(x_p) \cup W_u(x_r)) \cap f_c$ and $W_u(x_p) \cap f_c$ define local fibrations over $W_u(x_r) \cap f_c$ with the same fiber. Consequently $W_u(x_r) \cap f_c \subset \overline{W_u(x_p)} \cap f_c$ and we have

$$(W_u(x_p) \cup \{W_u(x_j) \mid x_j \in F_{x_p} \text{ and } j > r-1\}) \cap f_c \subset \overline{W_u(x_p)} \cap f_c. \tag{7.40}$$

Now consider a fixed neighborhood, Ω, of x_r. All trajectories of Φ from $\overline{W_u}(x_p) \cap f_{c'}$ which do not enter Ω are images via an invertible diffeomorphism $\eta(-s,\cdot)$ of some trajectories of $\overline{W_u}(x_p) \cap f_{c'}$, for $c_r < c' < c_{r+1}$. Since

$$\overline{W_u}(x_p) \cap f_{c'} = (W_u(x_p) \cup \{W_u(x_j) \mid x_j \in F_{x_p} \text{ and } c_j > c'\}) \cap f_{c'},$$

these trajectories are contained in

$$(W_u(x_p) \cup \{W_u(x_j) \mid x_j \in F_{x_p},\ j > r-1\}) \cap f_c. \tag{7.41}$$

The other orbits enter Ω. In Ω we may assume that the local picture is known as in (7.7). The only accumulation points in Ω which belong to $W_u(x_p) \cap f_c$ are in $W_u(x_r) \cap f_c$. These observations, together with (7.41), yield the reverse inclusion to (7.40). Thus Step 2 is also valid for this case.

Step 3. If (ii) is satisfied for $i = p$, $c_r < c < c_p$ with c a noncritical value, and (i) holds for $i = p$, $r+1 \le j \le p-1$, then (i) holds for $i = p$ and $r \le j \le p-1$.

We consider again the two cases of Step 2:

Case 1. $W_u(x_p) \cap W_s(x_r) = \phi$. Then $W_u(x_j) \cap W_s(x_r) = \phi$ for any $x_j \in F_{x_p}$, $j > r$ for otherwise $W_u(x_j)$ and $W_s(x_r)$ would intersect transversally strongly and uniformly. Since

$W_u(x_p)$ intersects $W_s(x_j)$ transversally strongly and uniformly, $W_u(x_j)$ is contained in $\overline{W_u(x_p)}$ and $W_u(x_p) \cup W_u(x_j)$ fibers locally over $W_u(x_j)$ with fiber $(W_u(x_j) \cap W_s(x_j)) \cup \{x_j\}$. This implies that $W_u(x_p) \cap W_s(x_r) \neq \phi$, contrary to our assumption. Indeed, let $y \in (W_u(x_p) \cap W_s(x_j)) \cup \{x_j\}$ in a given neighborhood of x_j with $y \neq x_j$. Let S_y be a section of the (trivializable) local bundle $W_u(x_p) \cup W_u(x_j) \to W_u(x_j)$ associated to y, i.e. modulo a trivialization chart, $S_y(x) = (x, y)$ for $x \in W_u(x_j)$. Then for y near x_j, $S_y(W_u(x_j))$ intersects $W_s(x_r)$ since the intersection of $W_u(x_j)$ and $W_s(x_r)$ is strong and uniform and since $S_y(W_u(x_j))$ is only a perturbation of $W_u(x_j)$. Any point in $S_y(W_u(x_j)) \cap W_s(x_r)$ lies in $W_u(x_j) \cap W_s(x_r)$. Hence our claim that $W_u(x_j) \cap W_s(x_r) = \phi$ for any $x_j \in F_{x_p}$, $j > r$ follows.

Since $x_r \notin F_{x_p}$ and $W_u(x_j) \cap W_s(x_r) = \phi$ for $j < r$, applying (i) which holds for j, $r \leq p-1$, we have

$$\mu\Bigg(\bigcup_{x_j \in F_{x_p}} W_u(x_j), W_s(x_r)\Bigg) \geq \rho > 0. \tag{7.42}$$

Consider an open neighborhood, $\mathcal{O}$, of $\bigcup_{x_j \in F_{x_p}} W_u(x_j)$. $\mathcal{O}$ may be chosen so that

$$\mu(\mathcal{O}, W_s(x_r)) \geq \frac{1}{2}\rho > 0. \tag{7.43}$$

We claim that for noncritical values $c \in (c_r, c_p)$,

$$(W_u(x_p) \backslash \mathcal{O}) \cap f_c \cap \overline{W}_s(x_r) = \phi. \tag{7.44}$$

Indeed $(W_u(x_p) \backslash \mathcal{O}) \cap f_c$ is a compact set. Clearly $\overline{W_s}(x_r) \subset \bigcup_{j \geq r} W_s(x_j)$. If (7.44) were false, we would have for some $j \geq r+1$

$$(W_u(x_p) \backslash \mathcal{O}) \cap f_c \cap W_s(x_j) \supset \{x\} \tag{7.45}$$

where $x \in \overline{W}_s(x_r)$. Then $x_j \in F_{x_p}$. $\mathcal{O}$ is a neighborhood of $W_u(x_j)$, and by (7.43), $W_s(x_r)$ does not meet $\mathcal{O}$. Observe that the decreasing orbit $\eta(-t, \cdot)$ starting at x enters $\mathcal{O}$ since $x \in W_s(x_j)$ and $x_j \in F_{x_p}$. Moreover $x \in \overline{W}_s(x_r) \backslash W_s(x_r)$ so in any neighborhood

of x we may find $y \in W_s(x_r)$, $y \notin W_s(x_j)$ with $\mu(y, W_s(x_j))$ as small as desired. If this neighborhood is small enough, then the decreasing orbit $\eta(-t,\cdot)$ starting at y will enter $\mathcal{O}$, i.e.

$$\eta(-t(y), y) \in \mathcal{O} \cap W_s(x_r), \tag{7.46}$$

a contradiction. Hence (7.44) holds and

$$\mu((W_u(x_p)\backslash\mathcal{O}) \cap f_c,\ \overline{W}_s(x_r)) > 0. \tag{7.47}$$

Now (7.47), (7.43), and (ii) imply

$$\mu(\overline{W}_u(x_p) \cap f_c, \overline{W}_s(x_r)) \geq \min\Big(\frac{\rho}{2}, \mu((W_u(x_p)\backslash\mathcal{O}) \cap f_c, \overline{W}_s(x_r))\Big). \tag{7.48}$$

Fixing $c \in (c_r, c_{r+1})$, we get a lower bound for $\mu(\overline{W}_u(x_p) \cap f_c, \overline{W}_s(x_r))$:

$$\mu(\overline{W}_u(x_p) \cap f_c, \overline{W}_s(x_r)) \geq \rho_1(c) > 0. \tag{7.49}$$

Consider the reduction given by the Morse Lemma in a neighborhood of x_r. This provides us with a description of the local behavior of the level sets of f for c close to c_r. We choose c so that this description is available. In local coordinates, the flow for Φ, $\eta(-t,\cdot)$, $t > 0$ increases the distance of initial points to $W_s(x_r)$ (for a suitable choice for this distance). Therefore

$$\mu(\eta(-t, \overline{W}_u(x_p) \cap f_c), W_s(x_r)) \geq \rho_1(c) > 0 \tag{7.50}$$

for all $t \geq 0$. In particular $\mu(\eta(-t, \overline{W}_u(x_p) \cap f_c), x_r) \geq \rho_1(c) > 0$ for all $t \geq 0$ and we may deform $\overline{W}_u(x_p)$ to any level $c' \in (c_{r-1}, c_r)$ using the flow of Φ. We modify the parametrization so that this deformation takes place within the fixed time $\bar{t} > 0$. For simplicity we keep the notation $\eta(-t,\cdot)$. Therefore for any $c' \in (c_{r-1}, c_r)$,

$$\eta(-\bar{t}, \overline{W}_u(x_p) \cap f_c) = \overline{W}_u(x_p) \cap f_{c'} \tag{7.51}$$

and

$$\mu(\overline{W}_u(x_p) \cap f_{c'}, W_s(x_r) \geq \rho_1(c). \tag{7.52}$$

Since $c' < c_r$, we also have

$$\mu(f^{c'}, W_s(x_r)) \geq \rho_2 > 0. \tag{7.53}$$

Hence

$$\mu(W_u(x_p), W_s(x_r)) \geq \inf(\rho_2, \rho_1(c)) > 0 \tag{7.54}$$

and (i) is proved for this case.

Case 2. $W_u(x_p) \cap W_s(x_r) \neq \phi$.

Since (ii) holds for $i = p$ and $c \in (c_r, c_{r+1})$, we have

$$\overline{W_u}(x_p) \cap f_c = (W_u(x_p) \cup \{W_u(x_j) \mid x_j \in F_{x_p} \text{ and } j > r\}) \cap f_c. \tag{7.55}$$

Consider an open neighborhood $\mathcal{O}$ of $\{W_u(x_j) \mid x_j \in F_{x_p}\}$. The intersections $W_u(x_j) \cap W_s(x_r)$, $j > r$ are transversal, strongly and uniformly. Therefore $\mathcal{O}$ can be chosen so small that the same is true for the part of $W_u(x_p)$ in $\mathcal{O}$ with respect to $W_s(x_r)$. Indeed since (i) holds for $i = p$ and $r + 1 \leq j \leq p - 1$, Proposition 7.24 and 7.31 hold for $W_u(x_p) \cap W_s(x_j)$ and we have the usual local bundle structure of $W_u(x_p)$ over $W_u(x_j)$ with the related tangent bundle property. Therefore any strong and uniform transversality property for $W_u(x_j)$ translates to $W_u(x_p)$. More precisely since the tangent bundle to $W_u(x_p)$, along neighborhoods of compact subsets of $W_u(x_j)$, contains a subbundle extending the tangent bundle to $W_u(x_j)$, the strong and uniform intersection of $W_u(x_j)$ and $W_s(x_r)$ implies that of $W_u(x_p)$ and $W_s(x_r)$ along neighborhoods of such compact subsets. If such neighborhoods are removed, by (ii) we are left with neighborhoods of sets of the type $W_u(x_k) \cap W_s(x_r)$, $k < j$, $x_k \in F_{x_j}$. Indeed (ii) implies that

$$\overline{W_u}(x_j) \cap f_c = (W_u(x_j) \cup \{W_u(x_k) \mid x_k \in F_{x_j}\}) \cap f_c.$$

Therefore we may choose a compact set in $\overline{W_u}(x_j)$ such that its complement is a suitable neighborhood of $\{W_u(x_k) \mid x_k \in F_{x_j}\} \cap f_c$ and our argument above applies. This lowers the index j and ultimately shows, when all possible indices are used, that $W_u(x_p)$ intersects $W_s(x_r)$ transversally strongly and uniformly in $\mathcal{O}$.

Consider then $(W_u(x_j) \backslash \mathcal{O}) \cap f^{-1}(c)$. This is compact and so is $W_s(x_r) \cap f^{-1}(c)$ where $c \in (c_r, c_{r+1})$. Along the boundary of $W_u(x_p) \backslash \mathcal{O}$, $W_u(x_p)$ is transverse to $W_s(x_r) \cap f^{-1}(c)$. By a standard perturbation argument, we can make $(W_u(x_p) \backslash \mathcal{O}) \cap f^{-1}(c)$ transverse to $W_s(x_r) \cap f^{-1}(c)$ everywhere by modifying the flow along the normal bundle of $W_s(x_r) \cap f^{-1}(c)$ in $f^{-1}(c)$ without affecting the boundary. The perturbed $W_u(x_p) \cap f^{-1}(c)$ is then transversal to $W_s(x_r) \cap f^{-1}(c)$. This transversality is strong and uniform since it was so in $\mathcal{O}$ and since on the sets $W_u(x_p) \backslash \mathcal{O}$ we are dealing with intersections along compact sets. Thus (i) is proved for Case 2 and the proof of Step 3 is complete.

The three steps together imply that (i)-(iii) are satisfied by (x_p, x_j) for $j < p$ and the proof of $2° - 4°$ is complete. Now we turn to the proof of $5°$. First we will establish that $W_u(a, b) \equiv \mathcal{D}$ is an ENR. Since $\mathcal{D} \subset \mathbf{R}^k$, by a theorem of Borsuk [12], it suffices to show that $\mathcal{D}$ is locally compact and locally contractible. By $4°$ of Theorem 7.2, at any point $z \in W_u(x)$, $\mathcal{D}$ locally is a bundle over $W_u(x)$ with fiber $\mathcal{F}_z$ homeomorphic to $(W_s(x) \cap \{W_u(y) \mid y > x\}) \cup \{x\}$. Note that $y > x$ means that $W_u(y) \cap W_s(x) \neq \phi$. The fiber $\mathcal{F}_z$ is contractible using the decreasing flow. Since $W_u(x)$ is a finite dimensional manifold, z has a contractible neighborhood N in $W_u(x)$. Therefore $\mathcal{D}$ is locally contractible at z, the contractible neighborhood of z in $\mathcal{D}$ being $N \times \mathcal{F}_z$ in a trivialization over N about z. Furthermore by $3°$ of Theorem 7.2, $\mathcal{D}$ is locally compact since $\overline{\mathcal{D}} = \mathcal{D}$. Hence $\mathcal{D}$ is an ENR.

To show that f^b retracts by deformation onto $\mathcal{D}$, let $b_1 < \cdots < b_m$ be the critical values of f between a and b and $x_1, \ldots, x_m$ the corresponding critical points. Each of these critical points admits a neighborhood of the type

$$|X|^2 + |Y|^2 \leq \epsilon_i \tag{7.56}$$

for some $\epsilon_i > 0$ where (X, Y) are local coordinates corresponding to a Morse Lemma reduction, i.e. X is the coordinate along the unstable manifold E^- and Y is the coordinate along the stable manifold E^+. We will also use $W_u(x_r)$ and $W_s(x_r)$ to denote E^- and E^+. Further smallness conditions will be imposed on ϵ_i later.

Let $\eta(-t, \cdot)$ denote the flow for Φ, $t \geq 0$. Consider the balls $B_1, \ldots, B_m$ in the X, Y coordinates around $x_1, \ldots, x_m$ respectively. Let

$$W_i = \bigcup_{t \geq 0} \eta(-t, B_i) \tag{7.57}$$

and

$$U(\epsilon_1, \ldots, \epsilon_m) \equiv U_\epsilon = \left(\bigcup_{i=1}^{m} W_i \right) \cup f^a. \tag{7.58}$$

Each set W_i is an ℓ dimensional manifold with boundary where $\ell = \dim \mathcal{M}$. This follows since the set

$$S_i^u = \{(X, Y) \mid |X|^2 + |Y|^2 = \epsilon_i, |X|^2 \geq |Y|^2\} \tag{7.59}$$

is a section for the flow for Φ which sweeps out W outside B_i via $\eta(-t, \cdot)$.

Set

$$W_i^u = W_i \backslash B_i. \tag{7.60}$$

Clearly W_i defines a tubular neighborhood of $W_u(x_i)$, and hence a fibration over $W_u(x_i)$, the fiber being diffeomorphic to the disk $D^+ = \{(0, Y) \mid |Y|^2 \leq \epsilon_i\}$. If $K_i \subset W_u(x_i)$ is compact, ϵ_i may be chosen so that the diameter of the fiber remains small along K_i. It is then easy to extend the tangent bundle to $W_u(x_i)$ to W_i along K_i. In fact, as in (7.27)-(7.28), we can assume that this extension has been carried out on a fixed neighborhood $U(K_i)$:

$$g_{1i} : F(K_i) \to U(K_i). \tag{7.61}$$

The fiber of g_{1i} is diffeomorphic to the tangent space to $W_u(x_i)$ and

$$(7.62) \qquad g_{1i}\Big|_{g_{1i}^{-1}(K_i)} : g_{1i}^{-1}(K_i) \to K_i$$

is the tangent bundle to $W_u(x_i)$ restricted to K_i. Now g_{1i} defines an orthogonal bundle

$$(7.63) \qquad g_{2i} : (F(K_i))^{\perp} \to U(K_i).$$

If we are given another vector bundle $G = G(K_i)$ which is a subbundle of the tangent bundle of $\mathcal{M}$ and which is defined in a set $\mathcal{O} \subset U(K_i)$, we can split G over $F(K_i) \oplus F(K_i)^{\perp}$. The following notation will be used for this situation. If $v \in \mathcal{O}$, $v_z \in (G(K_i))_z$ denotes a vector in the fiber of G at z and v_z can be written as $v_z = h_z + k_z$ where $h_z \in (F(K_i))_z$ and $k_z \in ((F(K_i))^{\perp})_z$.

To complete the proof of 5°, the following result is required. Here $\mathbf{R}^+$ denotes the positive reals.

Proposition 7.64. *There exist continuous functions* $\varphi_1, \ldots, \varphi_m$ *with* $\varphi_i : (\mathbf{R}^+)^{i-1} \to \mathbf{R}^+$ *such that if* $0 < \epsilon_i < \varphi_i(\epsilon_1, \ldots, \epsilon_{i-1})$ *for all* $i = 1, \ldots, m$, *then for any* p-*tuple* $(x_{i_1}, \ldots, x_{i_p})$, $b_{i_{j+1}} > b_{i_j}$, $j = 1, \ldots, p$, $1 \le p \le m$, *we have*

1° $\sup_{\epsilon \to 0} \sup_{x \in U_\epsilon} \inf\{\mu(x,y) \mid y \in f^a \cup \{W_u(x_i) \mid a < f(x_i) < b\}\} = 0.$

2° If $x_{i_r} \not\subset F_{x_{i_{r+1}}} = \{x_k \in \mathcal{K}(f) \mid W_u(x_{i_{r+1}}) \cap W_s(x_k) \neq \phi\}$ for some index j, then $\bigcap_{j=1}^{p} \partial W_{i_j} = \phi$,

3° If $x_{i_j} \in F_{x_{i_{j+1}}}$, $0 \le j+1 \le p$, then the sets (∂W_{i_j}), $1 \le j \le p$, intersect transversally. Hence the intersection $\bigcap_{j=1}^{p} \partial W_{i_j}$ is a manifold $\mathcal{M}(\epsilon, i_1, \ldots, i_p) \equiv \mathcal{M}(\epsilon, p)$. Furthermore its tangent bundle contains a subbundle $G(\epsilon, p)$ with the following property:

$$(7.65) \qquad \lim_{\epsilon_{i_1}, \ldots, \epsilon_{i_p} \to 0} \ \sup_{z \in \mathcal{M}(\epsilon,p) \cap f^{-1}(c)} \ \sup \frac{\|h_z\|}{\|k_z\|} = 0$$

and $\dim G_z = \dim W_u(x_{i_1}) = \dim(F(K_{i_1}))_z$. In (7.65) c denotes a noncritical value with $c \in (b_{i_1 - 1}, b_{i_1})$.

We delay the proof of Proposition 7.64 for now and complete the proof of 5°. By the definition of U_ϵ, its boundary is made up of pieces of $f^{-1}(a)$ and pieces of ∂W_i. With the aid of Lemma 7.64, the intersections of any number of these sets are transversal. Therefore such intersections are manifolds $\mathcal{M}(\epsilon, p)$. The closure of these manifolds may have a boundary, e.g. if $x_{i_0} \in F_{x_{i_1}}$, then $\mathcal{M}(\epsilon, i_0, i_1, \dots, i_p)$ is a boundary portion of some part of $\overline{\mathcal{M}(\epsilon, p)}$. On each manifold $\mathcal{M}(\epsilon, p)$, we may define an inward normal to U_ϵ as follows: on $\partial W_{i_1} \cap \cdots \cap \partial W_{i_p}$, the tangent planes are independent. Therefore they intersect transversally and define independent linear forms. We can pick one which points inwards for each of the ∂W_{i_j} and thus for ∂U_ϵ. The same procedure applies if $f^{-1}(a)$ is added. Since the set of inward normals is convex for each linear form, we may glue these normals continuously and thus, even though ∂U_ϵ is not a manifold, being made up of pieces of manifolds, we can continuously define a vector field v along ∂U_ϵ pointing inwards to U_ϵ:

$$v : \partial U_\epsilon \to T\mathcal{M} \tag{7.66}$$

$$z \to v_z$$

where v_z points inwards to U_ϵ. Using e.g. a tubular neighborhood of ∂U_ϵ, v may be extended to all of $\mathcal{M}$ with $v = 0$ outside of a given neighborhood of ∂U_ϵ.

To be precise, we require that $v \equiv 0$ in $U_{\epsilon'}$ where $\epsilon' = (\epsilon'_1, \dots, \epsilon'_m)$ is such that $\overline{U_{\epsilon'}} \subset \operatorname{int} U_\epsilon$. Since all of the critical points of f between levels a and b lie in the interior of $\overline{U_{\epsilon'}}$, there exists $\beta > 0$ such that

$$\|\Phi(x)\| \geq \beta > 0 \tag{7.67}$$

for all $x \in f^b \cap (\mathcal{M} \backslash U_{\epsilon'})$. Hence for θ small enough, $\Phi - \theta v$ is a pseudogradient vector field for f in $f^{-1}(a, b)$. We may choose $\theta > 0$ such that

$$f'(x)(\Phi(x) - \theta v(x)) \geq \frac{1}{2}\beta^2 \tag{7.68}$$

for all $x \in f^b \cap (\mathcal{M} \backslash U_{\epsilon'})$. Consider the decreasing flow $\varphi(t, \cdot)$ for $\Phi - \theta v$:

$$\frac{d\varphi}{dt}(-t, x) = \Phi(\varphi) - \theta v(\varphi), \quad \varphi(0, x) = x. \tag{7.69}$$

Given any $x \in f^{-1}(b)$, there exists a unique smallest $t(x)$ such that

$$\varphi(-t(x), x) \in \overline{U_\epsilon}. \tag{7.70}$$

Indeed $\varphi(-t, x)$ must enter $\overline{U_\epsilon}$ since:

$$f(\varphi(-t, x))' = -f'(\varphi)(\Phi(\varphi) - \theta v(\varphi)) \leq -\frac{\beta^2}{2} \tag{7.71}$$

if $\varphi(-t, x) \in f^b \cap (\mathcal{M} \setminus U_{\epsilon'})$. Furthermore since $-\Phi$ is either tangent to or points inwards to ∂U_ϵ and since θv strictly points inwards to ∂U_ϵ, the trajectory cannot escape $\overline{U_\epsilon}$ and $t(x)$ is unique and continuous. The deformation

$$\begin{gathered} D : [0,1] \times f^b \to f^b \\ D(\tau, x) = \varphi(-\tau t(x), x) \end{gathered} \tag{7.72}$$

retracts f^b by deformation onto U_ϵ.

In order to conclude the proof, we must show that U_ϵ retracts by deformation onto $\mathcal{D} = W_u(a, b)$. Since $\mathcal{D}$ is an ENR, the Cech homology of $\mathcal{D}$ and the singular homology of $\mathcal{D}$ coincide [10]. We established that f^b can be retracted by deformation onto U_ϵ, which is a subset of a neighborhood of $\mathcal{D}$. Since ϵ can be made arbitrarily small, it follows that the homology of f^b is isomorphic to the Cech homology of $\mathcal{D}$ and hence to its singular homology. The above argument in fact shows that the injection of $\mathcal{D}$ to f^b is a homotopy equivalence since the argument extends to homotopy groups [10]. Thus f^b and $\mathcal{D}$ have the same homotopy type. This is enough for the purpose of this paper. Establishing that this homotopy equivalence can be taken to be a retraction by deformation is more technical and we refer to [15] for this point.

Remark 7.73. Taking $c \in (a, b_1)$, the invariant set in the sense of C. Conley referred to in Remark 7.3 would then be

$$f^c \cup \{W_u(x) \mid x \in \mathcal{K}(f),\ \ a < f(x) \quad b\}.$$

We now conclude this section with *the somewhat lengthy:*

Proof of Proposition 7.64. Since the sets W_i are images by a decreasing flow map of the B_i's, 2° follows on showing that

$$W_{i_{r+1}} \cap B_{i_r} = \phi. \tag{7.74}$$

Since $W_u(x_{i_{r+1}}) \cap W_s(x_{i_r}) = \phi$, (i) implies that

$$\mu(W_u(x_{i_{r+1}}), W_u(x_{i_r})) \geq \rho_{r+1,r} > 0. \tag{7.75}$$

Thus

$$\mu(\overline{W}_u(x_{i_{r+1}}), \overline{W}_u(x_{i_r})) \geq \rho_{r+1,r} > 0. \tag{7.76}$$

Choosing

$$\epsilon_{i_r} < \frac{1}{4}\rho_{r+1,r}, \tag{7.77}$$

(7.74) follows from (7.76)-(7.77) if we can choose $\epsilon_{i_{r+1}}$ so that

$$\mu(x, \overline{W}_u(x_{i_{r+1}})) < \frac{1}{4}\rho_{r+1,r} \tag{7.78}$$

for all $x \in W_{i_{r+1}}$, or if

$$\lim_{\epsilon_{i_{r+1}} \to 0} \sup_{x_r \in W_{i_{r+1}}} \mu(x, \overline{W_u}(x_{i_{r+1}})) = 0. \tag{7.79}$$

Observe that (7.79) together with 3° of Theorem 7.2 implies 1° of Proposition 7.64. Hence 1° − 2° of the Proposition follow from (7.79). Now (7.79) is a consequence of 3° − 4° of Theorem 7.2 as will be shown next. The set W_i is obtained from B_i by using the flow $\eta(-t, \cdot)$ for $t \geq 0$. Consider a fixed ball $B^-(x_i, \rho)$, $\rho > 0$ in E^- about x_i. It is clear that for a suitable ρ independent of ϵ_i and any $x \in W_i$, $x = (X, Y)$ such that $X \in B^-(x_i, \rho)$, we have

$$\mu(x, E^-) = \mu(x, \overline{W}_u(x_i)) < \epsilon_i. \tag{7.80}$$

This is simply due to the local behavior of the flow $\eta(-t,\cdot)$ which contracts the Y-directions and expands the X-directions, i.e.

$$\lim_{\epsilon_i\to 0}\ \sup_{\substack{x=(X,Y)\in W_i\\ X\in B^-(x_i,\rho)}} \mu(x,\overline{W}_u(x_i))=0. \tag{7.81}$$

By (7.81), the same result holds on any compact set $K_i\subset W_u(x_i)$. Indeed such a set is covered by $\eta(-t(K_i),B^-(x_i,\rho))$ where $t(K_i)\in\mathbf{R}$ depends only on K_i and ρ. Hence for any $x\in W_i$ having $z\in K_i$ as base point in the fibration $W_i\to W_u(x_i)$ with fiber diffeomorphic to S_i^u:

$$\mu(x,z)\le\left(\sup_{\substack{t\in[0,t(K_i)]\\ y\in\mathcal{M}}}\|D\eta(-t,y)\|\right)\mu(\eta(t(K_i),x),\eta(t(K_i),z))<C\epsilon_i \tag{7.82}$$

where

$$C=\sup_{(t,y)\in[0,t(K_1)]\times\mathcal{M}}\|D\eta(-t,y)\|.$$

Thus we are left with those points x for which the base point z belongs to a neighborhood of $\overline{W}_u(x_i)\backslash W_u(x_i)$, i.e. of $\{W_u(x_j)\mid x_j\in F_{x_i}\}$. We may assume that we start with base points $z\in\partial K_i$, the boundary of a large ball in $W_u(x_i)$ and thus, using (7.82) with ϵ_i small enough since z belongs to a neighborhood of $\{W_u(x_j)\mid x_j\in F_{x_i}\}$, with points x near $\{W_u(x_j)\mid x_j\in F_{x_i}\}$. Our goal is to prove that such points remain close to $\{W_u(x_j)\mid x_j\in F_{x_i}\}=\{\overline{W}_u(x_j)\mid x_j\in F_{x_i}\}$ when they are subjected to the flow $\eta(-t,\cdot)$. Therefore we return to the situation we started with but with a lower index $j<i$. Using a decreasing induction, at the last step we arrive at a situation where $W_u(x_j)=\overline{W}_u(x_j)$; hence the result.

Now we will prove 3°. We claim the condition on the subbundle $G(\epsilon,p)$

$$\lim_{\epsilon_{i_1},\dots,\epsilon_{i_p}\to 0}\ \sup_{z\in\mathcal{M}(\epsilon,p)\cap f^{-1}(c)}\ \sup_{v_z\in G_z}\frac{\|k_z\|}{\|h_z\|}=0 \tag{7.83}$$

may be replaced by a similar one with the constraint $f(z)=c$ replaced by the requirement that $z=(X,Y)$, $|X|=\rho$, $|Y|\le\rho$ for a fixed ρ. To justify this, observe that the

coordinates (X, Y) are local Morse reduction coordinates about x_i. Since $c \in (b_{i_1-1}, b_{i_1})$ ρ may be chosen so that $f(z) > c$ if $z = (X, Y)$, $|X| \le \rho$, $|Y| \le \rho$. We know that as $\epsilon_{i_1} \to 0$,

$$\sup_{x \in W_{i_1}} \mu(x, W_{u_{i_1}}) \to 0.$$

Therefore we may also assume that for any $z \in \mathcal{M}(\epsilon, p)$ which may be written in the (X, Y) coordinates, we have $|Y| \le \rho$. For ϵ_{i_1} small enough and $z \in \mathcal{M}(\epsilon, p) \cap f^{-1}(c)$, the amount of time t, needed by $\eta(t, z)$ to reach $\{(X, Y) \mid |X| = \rho, \ |Y| \le \rho\}$ is bounded from above by a constant $C = C(\rho, c)$. Conversely if $z' \in \mathcal{M}(\epsilon, p)$, $z' = (X', Y')$, $|X'| = \rho$, $|Y'| \le \rho$, the amount of time, $-t$, needed by $\eta(-t, z')$ to reach $f^{-1}(c)$ is bounded from below by a constant $-C(\rho, c)$. The map $D\eta(t, \cdot)\Big|_{W_u(x_{i_1})}$ leaves the tangent space to $W_u(x_{i_1})$ invariant. Since any point $x \in W_{i_1} \cap f_c$ is close to $W_u(x_i)$ and the time t needed to get from $f^{-1}(c)$ to $z' = (X', Y')$, $|X'| = \rho$, $|Y'| \le \rho$ is globally bounded from above, we may replace (7.83) by

$$\lim_{\epsilon_{i_1}, \dots, \epsilon_{i_p} \to 0} \ \sup_{\substack{z \in \mathcal{M}(\epsilon, p), \\ z = (X, Y) \text{ near } x_{r_1}; |X| = \rho, |Y| \le \rho}} \ \sup_{v_z \in G_z} \frac{\|k_z\|}{\|h_z\|} = 0. \tag{7.84}$$

Now 3° will be proved by induction on p. For $p = 1$, ∂W_{i_1} is a manifold and we seek a subbundle of the tangent bundle to ∂W_{i_1} which satisfies (7.84). Choosing $\epsilon_{i_1} < \rho$, let

$$\partial S^u_{i_1} = \{(X', Y') \text{ near } x_{i_1} \mid |X'| = |Y'|, |X'|^2 + |Y'|^2 = \epsilon_{i_1}\}. \tag{7.85}$$

The tangent space to ∂W_{i_1} at a point $(X', Y') \in \partial S^u_{i_1}$, is the tangent space to this sphere, i.e.

$$\{(h_1', k_1') \mid X' \cdot h_1' + Y' \cdot k_1' = 0\}. \tag{7.86}$$

Consider a point $(X, Y) \in \partial W_{i_1}$ such that

$$|X| = \rho, |Y| \le \rho. \tag{7.87}$$

In fact $|Y| \leq \epsilon_{i_1} < \rho$. Such a point belongs to ∂W_{i_1} if and only if

$$e^{-t}\rho = e^{-t}|X| = e^{t}|Y| = \left(\frac{1}{2}\epsilon_{i_1}\right)^{1/2}, \tag{7.88}$$

i.e. $t = \frac{1}{2}\log\dfrac{2\rho^2}{\epsilon_{i_1}}$, $|Y| = \dfrac{\epsilon_{i_1}}{2\rho}$, or equivalently if

$$\eta\left(\frac{1}{2}\log\frac{2\rho^2}{\epsilon_{i_1}}, (X,Y)\right) \in \partial S^u_{i_1}. \tag{7.89}$$

The tangent space at (X,Y) is the image under $D\eta\left(-\frac{1}{2}\log\frac{2\rho^2}{\epsilon_{i_1}}, \cdot\right)$ of the tangent space at $(X',Y') = \eta\left(\frac{1}{2}\log\frac{2\rho^2}{\epsilon_{i_1}}, (X,Y)\right)$ to the sphere, i.e.

$$(h_1, k_1) = \left(\rho\left(\frac{2}{\epsilon_{i_1}}\right)^{1/2} h_1', \left(\frac{\epsilon_{i_1}}{2}\right)^{1/2}\rho^{-1}k_1'\right) \tag{7.90}$$

where

$$\frac{1}{\rho}\left(\frac{\epsilon_{i_1}}{2}\right)^{1/2} X \cdot h_1' + \rho\left(\frac{2}{\epsilon_{i_1}}\right)^{1/2} Y \cdot k_1' = 0$$

or equivalently

$$\{(h_1, k_1) \mid X \cdot h_1 + \frac{4\rho^4}{\epsilon_{i_1}^2} Y \cdot k_1 = 0\}. \tag{7.92}$$

Since

$$|X| = \rho; \qquad |Y| = \frac{\epsilon_{i_1}}{2\rho}, \tag{7.93}$$

for any $h_1 \in E^-$, the vector

$$h_1 - \rho^{-2}(X \cdot h_1)Y \tag{7.94}$$

belongs to the tangent space to ∂W_{i_1}, at (X,Y). By taking the image of these vectors by $D\eta(-t,\cdot)$, $t \geq 0$, (7.94) defines a subbundle of $G(\epsilon_{i_1}, i_1)$. Furthermore if $(X,Y) \in \partial W_i$, and $|X| = \rho$, then $|X| = \rho$, $|Y| = \frac{\epsilon_{i_1}}{2\rho}$ and setting

$$v_{(X,Y)} = v_z = h_1 - \rho^{-2}(X \cdot h_1)Y, \tag{7.95}$$

we get

$$\frac{\|k_z\|}{\|h_z\|} \leq C\rho^{-2}|X||Y| < C\rho^{-2}\frac{\epsilon_{i_1}}{2} \tag{7.96}$$

where C is an upper bound for the norm of the projection from $E^+ = W_s(x_{i_1})$ onto $(F(K_{i_1}))^\perp$. We do this in order to conform to the precise definitions (7.61)-(7.63). Otherwise we could simply take $(F(K_{i_1}))^\perp = E^+$ with a suitable scalar product, $K_{i_1} = \{(X,Y) \mid |X| = \rho,\ |Y| \leq \rho\}$. The inequality (7.96) implies (7.83) and (7.84) and hence the induction for $p = 1$.

Now we give the argument for arbitrary p. We start with $(x_{i_1}, \ldots, x_{i_p})$, $x_{i_k} \in F_{x_{i_{k+1}}}$, $\mathcal{M}(\epsilon, p)$ is a manifold, the intersection of the sets ∂W_{i_j}, $j = 1, \ldots, p$ is transversal, and the tangent bundle to $\mathcal{M}(\epsilon,p)$ contains a subbundle $G(\epsilon,p)$ satisfying (7.83) or (7.84) near x_{i_1}. Now we add another critical point $x_{i_0} \in F_{x_{i_1}}$ to this family. It has a related W_{i_0} and ϵ_{i_0}. We want to study

$$\mathcal{M}(\epsilon, x_{i_0}, \ldots, x_{i_p}) \equiv \mathcal{M}'(\epsilon,p) = \mathcal{M}(\epsilon,p) \cap \partial W_{i_0}. \tag{7.97}$$

We may always assure, without loss of generality, that there is no critical point z such that $z \in F_{x_{i_1}}$ and $x_{i_0} \in F_z$. Indeed, should such a z exist, we could take it to be our present x_{i_0}. Proceeding in this way, by induction on all the possible intermediate elements $z_1, \ldots, z_m = x_{i_0}$ between x_{i_1} and x_{i_0} ($z_i \in F_{z_{i+1}}$, z_i maximal in $F_{z_{i+1}}$), we would establish 3° of Proposition 7.64 for the whole chain of indices. The statement for the subchain $(i_0, \ldots, i_p)$ follows immediately.

Therefore, in the sequel, x_{i_0} is maximal in $F_{x_{i_1}}$, i.e. there is no z such that $z \in F_{x_{i_1}}$ and $x_{i_0} \in F_z$. Let (X,Y) be local Morse reduction coordinates near x_{i_0} and (X',Y') be Morse coordinates near x_{i_1}. Let $\rho > 0$ be given and let

$$K^0 \equiv W_u(x_{i_1}) \cap \{(0,Y) \mid |Y| = \rho\}. \tag{7.98}$$

Let

(7.99) $\quad \mathcal{O}(K^0)$ be a small closed neighborhood of K^0 which does not contain x_{i_0}.

Let

$$(7.100) \qquad (S_{i_0}^u)^+ = \{(X,Y) \mid |X| \le |Y|, |X|^2 + |Y|^2 = \epsilon_{i_0}\}$$

and let $K^1 \subset W_u(x_{i_1})$ be:

$$(7.101) \qquad K^1 = \{(X',0) \mid |X'| = \rho\}.$$

The set K^1 is a section to the flow restricted to $W_u(x_{i_1})$ and thus, for any $z \in K^0$, there exists $t(z) > 0$ such that:

$$(7.102) \qquad \eta(t(z), z) \in K^1$$

where $t(z)$ is bounded by a constant independently of z:

$$(7.103) \qquad t(z) \le C(\rho).$$

By continuity, we may assume that, if $\mathcal{O}(K^o)$ is compact and small enough, then there exists a compact set containing K^1, $\mathcal{O}(K^1)$, such that, for any $z \in \mathcal{O}(K^o)$, there exists a $t(z) < 1 + C(\rho)$ such that $\eta(t(z), z)$ belongs to $\mathcal{O}(K^1)$. It is not difficult to see that we may take

$$\mathcal{O}(K^1) = \{(X',Y') \mid |X'| = \rho, |Y'| \le \rho\}.$$

Since $\mathcal{O}(K^1)$ is then transverse to the flow, $t(z)$ is uniquely defined.

Our induction provides us with a subbundle $G(\epsilon_{i_1}, \ldots, \epsilon_{i_p}, x_{i_1}, \ldots x_{i_p})$ of the tangent space to $\mathcal{M}(\epsilon, p)$, with a fiber having the dimension of $W_u(x_{i_1})$, such that (7.84) holds on $\mathcal{O}(K^1)$. The tangent space to $\mathcal{M}(\epsilon, p)$ at $z \in \mathcal{M}(\epsilon, p) \cap \mathcal{O}(K^0)$ is the image under $D\eta(-t(z), \cdot)$ of the tangent space at $\eta(t(z), z)$ to $\mathcal{M}(\epsilon, p)$. Setting

$$G_1(\epsilon, p) = D\eta(-t(\cdot), \cdot)(G(\epsilon, \rho)),$$

then $G_1(\epsilon, p)$ is a subbundle of the tangent bundle to $\mathcal{M}(\epsilon, p)$ on $\mathcal{M}(\epsilon, p) \cap \mathcal{O}(K^0)$, the fiber of which has the dimension of $W_u(x_{i_1})$. $G(\epsilon, p)$ satisfies (7.84) on $\mathcal{O}(K^1)$;

$D\eta(t,\cdot)\mid_{W_u(x_{i_1})}$ leaves the tangent space to $W_u(x_{i_1})$ invariant; and $t(z)$ is bounded by from above $1+C(\rho)$. Therefore, $G_1(\epsilon,p)$ satisfies condition (7.84) on $\mathcal{M}(\epsilon,p)\cap\mathcal{O}(K^0)$ i.e. there exists an extension $F(\mathcal{O}(K^0))$ to $\mathcal{O}(K^0)$ of the tangent bundle to $W_u(x_{i_1})$ on K^0 such that any $v_z\in(G_1)_z$, the fiber of G_1 at $z\in\mathcal{M}(\epsilon,p)\cap\mathcal{O}(K^0)$, splits: $v_z=h_z+k_z$, $h_z\in(F(\mathcal{O}(K^0)))_z$, $k_z\in((F(\mathcal{O}(K^0)))^{\perp})_z$ with h_z and k_z satisfying (7.84) uniformly for z in $\mathcal{M}(\epsilon,\rho)\cap\mathcal{O}(K^0)$.

Since the intersection of $W_u(x_{i_1})$ and $W_s(x_{i_1})$ is transversal strongly and uniformly, if $\mathcal{O}(K^0)$ is small enough, there exists a $\sigma>0$ such that for all $z\in\mathcal{M}(\epsilon,p)\cap\mathcal{O}(K^0)$, the tangent space to $\mathcal{M}(\epsilon,p)$ at z contains a subspace of the form

$$\{(h_1,B_zh_1)\mid h_1\in E^-,B_z\in\mathcal{L}(E^-,E^+),\|B_z\|\le\sigma\} \tag{7.104}$$

with σ is independent of z. Here $E^-=W_u(x_{i_0})$ and $E^+=W_s(x_{i_0})$ so the subspace given by (7.104) is a graph over $W_u(x_{i_0})$.

Observe that if ϵ_{i_0} is small enough, any $z\in\mathcal{M}'(\epsilon,p)$ is such that $\eta(t,z)\in\mathcal{M}(\epsilon,p)\cap\mathcal{O}(K^0)$ for a suitable t. Indeed points in $\mathcal{M}(\epsilon,p)\cap\partial W_{i_0}$ are images under $\eta(-t,\cdot)$, $t\ge0$ of points in $\mathcal{M}(\epsilon,p)\cap(S^u_{i_0})^+$ and such points are clearly images of points in $\mathcal{M}(\epsilon,p)\cap\mathcal{O}(K^0)$. The tangent plane to ∂W_{i_0} at a point $z=(X,Y)\in\mathcal{M}(\epsilon,p)\cap(S^u_{i_0})^+$ is

$$\mathcal{T}_z=\{(h_1,k_1)\mid X\cdot h_1+Y\cdot k_1=0\}. \tag{7.105}$$

If $z=\eta(-t,(\tilde X,\tilde Y))$ and $\tilde z=(\tilde X,\tilde Y)\in\mathcal{M}(\epsilon,p)\cap\mathcal{O}(K^0)$, then

$$(X,Y)=(e^t\tilde X,e^{-t}\tilde Y),\quad t>0 \tag{7.106}$$

and by (7.104), the tangent space at z to $\mathcal{M}(\epsilon,p)$ contains a subspace of the type:

$$\mathcal{H}_z=\{(e^th_1,e^{-t}B_{\tilde z}h_1)\mid h_1\in E^-,\|B_{\tilde z}\|\le\sigma\}. \tag{7.107}$$

$\mathcal{O}(K^0)$ is fixed, compact and such that $x_{i_0}\notin\mathcal{O}(K_0)$; ϵ_{i_0} can then be chosen so small that t, in (7.106), is large uniformly for $z\in\mathcal{M}(\epsilon,p)\cap(S^u_{i_0})^+$. Observe that the flow

preserves $\mathcal{M}(\epsilon,p)$ and is transverse to ∂W_{i_0} on $(S^u_{i_0})^+\backslash\partial S^u_{i_0}$ (see (7.105)). Therefore, the intersection of these two sets is transverse on $(S^u_{i_0})^+\backslash\partial S^u_{i_0}$. On $\partial S^u_{i_0}$, (7.107) defines a vector space transverse to the tangent space to ∂W_{i_0}. Indeed $(e^t X, e^{-t}B_{\tilde z}X)$ is in this vector space, while, since $|X|=|Y|$,

$$(7.108)\qquad \begin{aligned}|X\cdot e^t X + Y\cdot e^{-t}B_{\tilde z}X| &= |e^t|X|^2 + e^t Y\cdot B_{\tilde z}Y|\\ &\geq e^t|X|^2 - e^{-t}\sigma|X|^2 \geq |X|^2(e^t-\sigma e^{-t}) > 0\end{aligned}$$

for t large enough. Thus, the intersection of $\mathcal{M}(\epsilon,\rho)$ and ∂W_{i_0} is transverse along $(S^u_{i_0})^+$ and is therefore globally transverse since any point of $\mathcal{M}(\epsilon,p)\cap\partial W_{i_0}$ is obtained from $\mathcal{M}(\epsilon,p)\cap(S^u_{i_0})^+$ using the flow.

We now define $G(\epsilon_{i_0},\dots,\epsilon_{i_p},x_{i_0},\dots,x_{i_p})$ and prove the second part of 3° in the form (7.84). By definition, the fiber (G_z) at $z\in\mathcal{M}'(\epsilon,p)\cap(S^u_{i_0})^+$ is:

$$(7.109)\qquad G_z = \text{ orthogonal projection on } \mathcal{T}_z \text{ of } (\mathcal{H}_z\oplus\mathbf{R}(X,-Y)).$$

Observe that $\mathcal{H}_z\oplus\mathbf{R}(X,-Y)$ is a direct sum: $(X,-Y)$ cannot belong to $\mathcal{H}_z$ because $|Y|\geq|X|$ on $(S^u_{i_0})^+$ and because t in (7.106) - (1.107) is very large. Furthermore, as we have seen above, $\mathcal{H}_z\oplus\mathbf{R}(X,-Y)$ is transverse to $\mathcal{T}_z$ which is the tangent space to ∂W_{i_0} at $z\in(S^u_{i_0})^+$. Extending G to $\mathcal{M}'(\epsilon,p)$ via $D\eta(-\tau,\cdot)$, $\tau\geq 0$, we obtain a subbundle of the tangent bundle to $\mathcal{M}'(\epsilon,p)$ of dimension

$$(7.110)\qquad \dim(G(\epsilon_{i_0},\dots,\epsilon_{i_p},x_{i_0},\dots,x_{i_p}))_z = \dim E^- + 1 - 1 = \dim W_u(x_{i_0}).$$

G_z may be expressed in another way if $z\in\mathcal{M}(\epsilon,p)\cap\partial S^u_{i_0}$. Setting $z=(X,Y)=\eta(-t,\tilde z)$, $\tilde z=(\tilde X,\tilde Y)$ in $\mathcal{M}(\epsilon,p)\cap\mathcal{O}(K^0)$, we introduce:

$$(7.111)\qquad \mathcal{H}'_z = \{(e^t(h_1-\nu X), e^{-t}B_{\tilde z}(h_1-\nu X)) \mid h_1\in E^-\}$$

where

$$\nu = \frac{e^t X\cdot h_1 + e^{-t}B_{\tilde z}h_1\cdot Y}{e^t|X|^2 + Y\cdot e^{-t}B_{\tilde z}X}$$

Observe that $\mathcal{H}'_z$ is tangent to $\mathcal{M}'(\epsilon, p)$ since it satisfies (7.105) - (7.107) G_z may then be described as:

$$G_z = \mathcal{H}'_z \oplus \mathbf{R}(X, -Y) \tag{7.112}$$

since $(X, -Y)$ is tangent to $\mathcal{M}'(\epsilon, p)$ along $\partial S^u_{i_0}$. In order to prove (7.84), we observe again that

$$\mathcal{M}'(\epsilon, p) \cap \{(X, Y) \text{ near } x_{i_0} \mid |X| = \rho, |Y| \leq \rho\}$$

is the image under η of $\mathcal{M}'(\epsilon, p) \cap \partial S^u_{i_0}$, the time τ needed being very large, $\tau \to \infty$ as $\epsilon_{i_0} \to 0$. We set

$$\begin{cases} z_1 = (e^\tau X, e^{-\tau} Y), (X, Y) \in \mathcal{M}'(\epsilon, p) \cap \partial S^u_{i_0} \\ z_1 \in \mathcal{M}'(\epsilon, p), \quad e^\tau |X| = \rho, |X| = |Y|, |X|^2 + |Y|^2 = \epsilon_{i_0} \end{cases} \tag{7.113}$$

The fiber of $G(\epsilon_{i_0}, \ldots, \epsilon_{i_p}, x_{i_0}, \ldots, x_{i_p})$ at z_1 is

$$(e^{\tau+t}(h_1 - \nu X), e^{-(\tau+t)} B_{\bar{z}}(h_1 - \nu X)) \oplus \mathbf{R}(e^\tau X, -e^{-\tau} Y) \tag{7.114}$$

where

$$\tau + t \geq \tau = \log \frac{\rho}{|X|} = \log \sqrt{\frac{2}{\epsilon_{i_0}}} \rho \longrightarrow \infty \text{ as } \epsilon_{i_0} \to \infty. \tag{7.115}$$

Equation (7.114) can be written as

$$(e^{\tau+t} h_1, e^{-(\tau+t)} B_{\bar{z}}(h_1 - \nu X) - e^{t-\tau} \nu Y) \tag{7.116}$$

since (7.116) defines a space of dimension equal to dim $W_u(x_{i_0})$ contained in $G(\epsilon_{i_0}, \ldots, \epsilon_{i_p}, x_{i_0}, \ldots, x_{i_p})$. Since $|X| = |Y|$, (7.108) and (7.111) imply:

$$| \nu |X| + \nu |Y| | \leq 3 \|h_1\| \tag{7.117}$$

for ϵ_{i_0} small enough. Therefore if $v_z = (e^{\tau+t} h_1, e^{-(\tau+t)} B_{\bar{z}}(h_1 - \nu X) - e^{t-\tau} \nu Y)$, and if we take, for the sake of simplicity, $E^- \oplus E^+$ for the decomposition of $v_z = h_z + k_z$, we have

$$\frac{\|k_z\|}{\|h_z\|} \leq \frac{\|e^{-(\tau+t)} B_{\bar{z}}(h_1 - \nu X) - e^{t-\tau} \nu Y\|}{e^{\tau+t} \|h_1\|} \tag{7.118}$$

$$\leq C \frac{e^t}{e^{\tau+t}} \frac{\|h_1\|}{\|h_1\|} \leq C e^{-\tau} \to 0$$

uniformly as $\epsilon_{i_0} \to 0$. As can easily be seen, (7.84) can be written with such a splitting so (7.84) holds for $G(\epsilon_{i_0}, \ldots, \epsilon_{i_p}, x_{i_0}, \ldots x_{i_p})$. Thus the induction is established and the proofs of Proposition 7.64 and Theorem 7.2 are complete.

§8. The Retraction Theorem and Related Results: The infinite dimensional case.

In this section, the results of §7 will be extended so as to apply to the functional $\tilde{I}$ introduced in §4. In the process a set of *critical points at infinity* for $\tilde{I}$ will be introduced and its stable and unstable manifolds relative to the pseudogradient flow generated by $\tilde{Z}$ will be characterized.

The notation of §4 will be used freely here. Again $\mathcal{K}(\tilde{I})$ denotes the set of critical points of $\tilde{I}$, etc. We henceforth take ϵ_1 as in Proposition 2.9 and assume

$$(8.1) \qquad \beta_1 < \min\{|b-c| \; |b \neq c \in \tilde{I}(\mathcal{K}(\tilde{I})) \cup (\bigcup_{i\neq j=1}^{3} J_{ij}(\mathcal{K}(J_{ij})) \text{ and } b,c \leq M+1\}.$$

Also let $W_u(q_1,q_2)$, $W_s(q_1,q_2)$ denote the stable and unstable manifolds corresponding to $(q_1,q_2) \in \mathcal{K}(J_{12})$, etc.

Theorem 8.2. Let $\tilde{I}$ be as defined in §4. Then

1° Any trajectory of (4.23) with $q(0) \in \tilde{I}^{M+1}$ which does not enter $\tilde{I}^{\epsilon_1} = I^{\epsilon_1}$ or does not converge to a critical point of $\tilde{I}$ has a limit. The set of limits, $\mathcal{H}$, can be written as

$$\mathcal{H} = \bigcup_{i\neq j=1}^{3} \mathcal{H}_{ij}$$

where

$$\mathcal{H}_{ij} = \bigcup_{(\overline{q}_i,\overline{q}_j)\in\mathcal{K}_{ij}^{M+1}} \mathcal{H}_{ij}(\overline{q}_i,\overline{q}_j).$$

2° In the (q_1,q_2,Q_3) coordinates,

$$\mathcal{H}_{12}(\overline{q}_1,\overline{q}_2) = \{(\overline{q}_1,\overline{q}_2)\} \times \{Q_3 \in \mathbf{R}^{\ell} \mid |Q_3 - \frac{1}{2}[\overline{q}_1+\overline{q}_2]| \geq (\frac{1}{\beta_1}-1)^{1/2}\}.$$

where $(\overline{q}_1,\overline{q}_2) \in \mathcal{K}_{12}^{M}$.

3° $\mathcal{H}_{12}(\overline{q}_1,\overline{q}_2)$ possesses a Frehbolm stable manifold, $W_s^{\infty}(\overline{q}_1,\overline{q}_2)$, and a finite dimensional unstable manifold, $W_u^{\infty}(\overline{q}_1,\overline{q}_2)$. These manifolds can be characterized in (q_1,q_2,Q_3) coordinates as:

a. $W_u^\infty(\overline{q}_1, \overline{q}_2)\backslash\mathrm{int}\ I^{\epsilon_1}$ is a bundle over $W_u(\overline{q}_1, \overline{q}_2)\backslash\mathrm{int}\ J_{12}^{\epsilon_1-\beta_1}$ with fiber $F_{(q_1,q_2)}$ over (q_1, q_2) given by

$$F_{(q_1,q_2)} = \left\{ Q_3 \in \mathbf{R}^\ell \mid |Q_3 - \frac{1}{2}[q_1 + q_2]| \geq \left(\frac{1}{\beta_1} - 1\right)^{1/2} \right\}.$$

b. Let $c = J_{12}(\overline{q}_1, \overline{q}_2)$ and $0 < \epsilon < \frac{1}{2}\beta(c_1)$. Then

$$W_s^\infty(\overline{q}_1, \overline{q}_2) \cap \tilde{I}^{c+\epsilon} = \bigcup_{0 \leq \alpha \leq \epsilon} G_{s,\alpha}^\infty(\overline{q}_1, \overline{q}_2)$$

where $G_{s,\alpha}^\infty(\overline{q}_1, \overline{q}_2)$ fibers over $W_s(\overline{q}_1, \overline{q}_2) \cap J_{12}^{c+\alpha}$ with fiber at (q_1, q_2) given by

$$G_\alpha(q_1, q_2) = \left\{ Q_3 \in W^{1,2} \mid \frac{1}{2}\int_0^1 |\dot{Q}_3|^2 dt + \frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]|^2} \leq \epsilon - \alpha \right\}.$$

4° 2° and 3° hold for any $\mathcal{H}_{ij}$.

5° In any C^1 neighborhood of $\tilde{I}'$, there exists a perturbation L of $\tilde{I}'$ possessing the following properties:

a. $L = \tilde{Z}$ in $\mathcal{V}_2(i, j)$, $i \neq j$.

b. L is a pseudogradient vector field for $\tilde{I}$.

c. For any $q \in \mathcal{K}^{M+1}$ and $(\overline{q}_i, \overline{q}_j) \in \mathcal{K}_{ij}^{M+1}$, the stable manifold $W_s^\infty(\overline{q}_i, \overline{q}_j; L)$ for L through $(\overline{q}_i, \overline{q}_j)$ is transverse to $W_u(q; L)$.

d. For any $\hat{q} \in \mathcal{K}^{M+1}\backslash\{q\}$, $W_u(q; L)$ is transverse to $W_s(\hat{q}; L)$.

e. For any $(\hat{q}_i, \hat{q}_j) \in \mathcal{K}_{ij}^{M+1}\backslash\{(\overline{q}_i, \overline{q}_j)\}$, $W_u^\infty(\overline{q}_i, \overline{q}_j; L)$ is transverse to $W_s^\infty(\hat{q}_i, \hat{q}_j; L)$.

6° The set $\tilde{I}^{M+1}$ retracts by deformation onto

$$\mathcal{W}^\infty \cup \mathcal{V}_\epsilon(\mathcal{D}_{M+1})$$

where

$$\mathcal{D}_{M+1} = \bigcup_{q \in \mathcal{K}^{M+1}} W_u(q), \quad \mathcal{D}_{M+1}^\infty = \bigcup_{i \neq j = 1}^{3} \bigcup_{(\overline{q}_i, \overline{q}_j) \in \mathcal{K}_{ij}^{M+1}} W_u^\infty(\overline{q}_i, \overline{q}_j),$$

$$I^{\epsilon_1} \cup \mathcal{D}_{M+1}^\infty \subset \mathrm{int}\ \mathcal{W}^\infty \subset \mathcal{W}^\infty \subset \mathcal{V}_2,$$

$\partial\mathcal{W}^\infty$ is piecewise smooth, and $\mathcal{V}_\epsilon(\mathcal{D}_{M+1})$ is a small neighborhood of $\mathcal{D}_{M+1}$. Moreover

a. $\mathcal{W}^\infty$ retracts by deformation onto $I^{\epsilon_1} \cup \mathcal{D}^\infty_{M+1}$,

b. $\mathcal{V}_\epsilon(\mathcal{D}_{M+1})$ and $\mathcal{V}_\epsilon(\mathcal{D}_{M+1}) \cap \mathcal{W}^\infty$ are ANR's.

c. $\mathcal{D}_{M+1}$ is an ENR of dimension m and the homologies of $\mathcal{V}_\epsilon(\mathcal{D}_{M+1})$ and $\mathcal{V}_\epsilon(\mathcal{D}_{M+1} \cap \mathcal{W}^\infty)$ vanish in dimension larger than m.

Remark 8.3. (i) On the basis of 5°, we can assume $\tilde{Z} = L$ and our original stable and unstable manifolds $W_u(q), W_s(q), W^\infty_u(q)$, etc. have transversal intersections. (ii) A stronger result than 6° is valid: $\tilde{I}^{M+1}$ retracts by deformation onto $I^{\epsilon_1} \cup \mathcal{D}^\infty_{M+1} \cup \mathcal{D}_{M+1}$. See Bahri [15].

Proof of Theorem 8.3. Let us recall the construction of $\tilde{Z}$ given in §4, in particular (4.15)-(4.23). First Z_{12} was chosen to be a pseudogradient vector field for J_{12} such that the stable and unstable manifolds for Z_{12} in the region $\epsilon_1 \leq J_{12}(q) \leq M+1$ have a transversal intersection. Points on the unstable manifold between levels ϵ_1 and $M+1$ satisfy (4.15). Then $\tilde{Z}$ was defined in (4.20)-(4.22) as follows:

$$\frac{d}{d\tau}(q_1, q_2, q_3) = -\omega_{12} Y_{12} - (1-\omega_{12})\tilde{I}' \equiv -\tilde{Z} \tag{8.4}$$

where Y_{12} expressed in the (q_1, q_2, Q_3) coordinates in $\mathcal{V}_2(1,2)$ is given in component form by

$$\begin{aligned}
&(i) \quad \frac{d}{d\tau}(q_1, q_2) = -Z_{12}(q_1, q_2) \\
&(ii) \quad \frac{d}{d\tau}(Q_3 - [Q_3]) = -(Q_3 - [Q_3]) \\
&(iii) \quad \frac{d}{d\tau}([Q_3 - \frac{1}{2}(q_1+q_2)]) = (1-\tilde{\omega}_{12})\frac{[Q_3 - \frac{1}{2}(q_1+q_2)]}{|[Q_3 - \frac{1}{2}(q_1+q_2)]|}.
\end{aligned} \tag{8.5}$$

Now we will prove 1° – 2° of Theorem 8.2. We first want to describe the critical points at infinity of $\tilde{I}$, i.e. by Corollary 4.42, *the limits of decreasing trajectories* for (4.23) which remain in $\mathcal{V}_1(1,2)$ for large τ. On $\mathcal{V}_1, \tilde{Z} = Y_{12}$. Therefore we are dealing with (4.21). Let $q(\tau) = (q_1(\tau), q_2(\tau), Q_3(\tau))$ be a solution of (8.5) for $\tau > 0$ such that $\epsilon_1 \leq \tilde{I}(q(\tau)) \leq M$. We claim $(q_1(\tau), q_2(\tau))$ converges to a critical point $(\overline{q}_1, \overline{q}_2)$ of J_{12} as $\tau \to \infty$ provided that

$J_{12}(q_1(\tau), q_2(\tau)) \not\to 0$. Since on $\mathcal{V}_1$,

$$\tilde{I}(q) = J_{12}(q_1, q_2) + \frac{1}{2}\int_0^1 |\dot{Q}_3|^2 dt + \frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]|^2}, \tag{8.6}$$

we have

$$J_{12}(q_1(\tau), q_2(\tau)) \geq \tilde{I}(q(\tau)) - \frac{\beta(C_1)}{2} \geq \epsilon_1 - \frac{\beta(C_1)}{2}. \tag{8.7}$$

Choosing $\beta(C_1) < \epsilon_1$, (8.7) implies

$$J_{12}(q_1(\tau), q_2(\tau)) \geq \frac{1}{2}\epsilon_1 \tag{8.8}$$

for all $\tau > 0$ such that $q(\tau) \in \mathcal{V}_1$. Hence as $\tau \to \infty$, $(q_1(\tau), q_2(\tau))$ converges to a critical point $(\overline{q}_1, \overline{q}_2)$ of J_{12} with

$$\frac{\epsilon_1}{2} \leq J_{12}(\overline{q}_1, \overline{q}_2) \leq M + 1. \tag{8.9}$$

By (8.5) (ii),

$$\|\dot{Q}_3\|_{L^2}(\tau) \to 0 \tag{8.10}$$

as $\tau \to \infty$. Since $(q_1(\tau), q_2(\tau)) \to (\overline{q}_1, \overline{q}_2)$, for large τ we have

$$\sum_{i=1}^{2} \|q_i(\tau) - \frac{1}{2}[q_1 + q_2](\tau)\|_{L^\infty} \leq \frac{C_1}{4}. \tag{8.11}$$

Now consider (8.5) (iii). It shows

$$|[Q_3 - \frac{1}{2}(q_1 + q_2)](\tau)|^2$$

is nondecreasing as $\tau \to \infty$. Combining this observation with (8.10) implies

$$\frac{1}{2}\int_0^1 |\dot{Q}_3|(\tau)^2 dt + \frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)](\tau)|^2} \to \mu \geq 0$$

as $\tau \to \infty$. We claim $\mu > 0$, for if $\mu = 0$, then for large τ,

$$\frac{1}{2}\int_0^1 |\dot{Q}_3|^2(\tau) dt + \frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]^2(\tau)|} < \beta_1. \tag{8.12}$$

Now (8.11)-(8.12) imply that $(q_1, q_2, Q_3)(\tau) \in \mathcal{V}_0$ for large τ. Therefore $(1-\tilde{\omega}_{12})(q(\tau))) = 0$ for large τ and (8.5) (iii) implies

$$\frac{d}{d\tau}([Q_3 - \frac{1}{2}(q_1 + q_2)]) = 0 \tag{8.13}$$

for large τ, contrary to $\mu = 0$. Since $\mu > 0$, (8.12) shows we have

$$|w(\tau)| \equiv |[Q_3 - \frac{1}{2}(q_1 + q_2)](\tau)| \leq \left(\frac{1}{\mu} - 1\right)^{1/2} \tag{8.14}$$

for large τ. By (8.5) (iii),

$$\frac{1}{2}\frac{d}{d\tau}|w(\tau)|^2 = (1 - \tilde{\omega}_{12})|w|(\tau). \tag{8.15}$$

Since $q(\tau) \in \mathcal{V}_1$, we have

$$\frac{1}{1+|w|^2}(\tau) < \frac{\beta(C_1)}{2} \tag{8.16}$$

for large τ. Thus if $\beta(C_1) < 2$,

$$|w|(\tau) \geq \left(\frac{2}{\beta(C_1)} - 1\right)^{1/2} \equiv \Theta_1 > 0. \tag{8.17}$$

Using (8.15), we get

$$|w|^2(\tau) \geq \Theta_1^2 + \Theta_1 \int_{\tau_0}^{\tau} (1 - \tilde{\omega}_{12}) d\sigma \tag{8.18}$$

for $\tau \geq \tau_0$ and τ_0 large. Combining (8.14) and (8.18) yields

$$\int_{\tau_0}^{\infty} (1 - \tilde{\omega}_{12}) d\sigma < \infty. \tag{8.19}$$

Hence by (8.5) (iii),

$$\int_{\tau_0}^{\infty} |\frac{d}{d\tau} w| d\tau < \infty. \tag{8.20}$$

Therefore $w(\tau) = [Q_3 - \frac{1}{2}(q_1 + q_2)](\tau)$ converges to a limit w_∞ as $\tau \to \infty$. Clearly $1 - \tilde{\omega}_{12}(q(\tau))$ then converges to 0. Therefore

$$\mu = \frac{1}{1+|w_\infty|^2} \leq \beta_1 \tag{8.21}$$

via (4.16). Lastly observe that by Proposition 2.9′, any critical point of J_{12} satisfying (8.9) will satisfy

$$\epsilon_1 \leq J_{12}(\overline{q}_1, \overline{q}_2) \leq M + 1 \tag{8.22}$$

if ϵ_1 is small enough. Now (8.9)-(8.11), (8.20)-(8.22) imply that the limits as $\tau \to \infty$ of the trajectories we are studying lie in $\mathcal{H}_{12}$ where $\mathcal{H}_{12}$ was defined in $1° - 2°$ of Theorem 8.2. Conversely, any $q \in \mathcal{H}_{12}$ is the limit as $\tau \to \infty$ of an orbit of (8.5), namely the one with initial data q. This proves $1° - 2°$ of Theorem 8.2.

Now we study the "stable" and "unstable" manifolds $W_s^\infty(\overline{q}_1, \overline{q}_2)$ and $W_u^\infty(\overline{q}_1, \overline{q}_2)$ corresponding to points in $\mathcal{H}_{12}$. We begin with $W_u^\infty(\overline{q}_1, \overline{q}_2)$\int I^{ϵ_1}.

(8.23)$W_u^\infty(\overline{q}_1, \overline{q}_2)$\int $I^{\epsilon_1} q(\tau) = \{(q_1(\tau), q_2(\tau), Q_3(\tau)) \mid q(\tau)$ is a solution of (8.4) whose limit set as $\tau \to -\infty$ has a nonempty intersection with $\mathcal{H}_{12}$ and $I(q(\tau)) \geq \epsilon_1$ for all $\tau < 0\}$.

Arguing as in the proof of Corollary 4.42, there is a τ_0 so that if $\tau \leq \tau_0$, $q(\tau) \in \mathcal{V}_1$. For such τ, $q(\tau)$ may be expressed in (q_1, q_2, Q_3) coordinates and (8.5) holds for $\tau \leq \tau_0$. When $\tau \in (-\infty, \tau_0)$ decreases to $-\infty$, (8.5) shows that $\|\dot{Q}_3\|_{L^2}(\tau)$ is nondecreasing while $Q_3 - [Q_3]$ tends to 0 as $\tau \to -\infty$. Hence

$$\|\dot{Q}_3\|_{L^2}(\tau) = 0 \tag{8.24}$$

for $\tau \leq \tau_0$. Similarly $|[Q_3 - \frac{1}{2}(q_1 + q_2)]|(\tau)$ is nonincreasing as $\tau \to -\infty$ in $(-\infty, \tau_0)$ while

$$|[Q_3 - \frac{1}{2}(q_1 + q_2)]|(-\infty) > \left(\frac{1}{\beta_1} - 1\right)^{1/2}$$

so

$$\frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]|^2(\tau)} < \beta_1 \tag{8.25}$$

for $\tau \leq \tau_0$. Thus

$$\frac{1}{2}\int_0^1 |\dot{Q}_3|^2(\tau)dt + \frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)(\tau)]|^2} < \beta_1 \tag{8.26}$$

for $\tau \leq \tau_0$. On the other hand, $(q_1(\tau), q_2(\tau)) \in W_u(\overline{q}_1, \overline{q}_2)\backslash\text{int } J_{12}^{\epsilon_1 - \beta_1}$, the unstable manifold of Z_{12} at $(\overline{q}_1, \overline{q}_2)$. Consequently

$$\sum_{i=1}^{2} \|q_i - \frac{1}{2}[q_1 + q_2](\tau)\|_{L^\infty} \leq \frac{C_1}{4} \tag{8.27}$$

for $\tau \leq \tau_0$ by (4.15) and the choice of Z_{12}, C_1. (See (4.17) and the following paragraph.) Inequalities (8.26)-(8.27) imply that $q(\tau) \in \mathcal{V}_1$ for $\tau \leq \tau_1$ where $\tau_1 > \tau_0$. Therefore $\tau_0 = \infty$, i.e. if $q(\tau) \in W_u^\infty(\overline{q}_1, \overline{q}_2)\backslash\text{int } I^{\epsilon_1}$, then (8.25) holds for $\sigma \in (-\infty, \infty)$ and

$$(q_1, q_2) \in W_u(\overline{q}_1, \overline{q}_2)\backslash\text{int } J_{12}^{\epsilon_1 - \beta_1}, \tag{8.28}$$

$$\|\dot{Q}_3\|_{L^\infty}(\tau) = 0, \ \text{ i.e. } \ Q_3(\tau) = [Q_3](\tau) \in \mathbf{R}^\ell, \tag{8.29}$$

$$\frac{1}{1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]^2(\tau)} < \beta_1 \tag{8.30}$$

i.e.

$$|[Q_3 - \frac{1}{2}(q_1 + q_2)]|(\tau) > \left(\frac{1}{\beta_1} - 1\right)^{1/2}.$$

Therefore $W_u^\infty(\overline{q}_1, \overline{q}_2)\backslash\text{int } I^{\epsilon_1}$ fibers over $W_u(\overline{q}_1, \overline{q}_2)\backslash\text{int } J_{12}^{\epsilon - \beta_1}$ with fiber $F_{(q_1, q_2)}$ as stated in $3°a$ of Theorem 8.2. Furthermore since $\tilde{w}_{12} = 0$ in $\mathcal{V}_0$, the flow [illegible] ed to $W_u^\infty(\overline{q}_1, \overline{q}_2)\backslash\text{int } I^{\epsilon_1}$ has a nice representation:

$$\begin{aligned} &\frac{d}{d\tau}(q_1, q_2)(\tau) = -Z_{12}(q_1, q_2)(\tau) \\ &Q_3(\tau) = [Q_3](\tau) \in \mathbf{R}^\ell \\ &[Q_3 - \frac{1}{2}(q_1 + q_2)](\tau) \equiv \text{constant independent of } \tau. \end{aligned} \tag{8.31}$$

These equations show each orbit in $W_u^\infty(\overline{q}_1, \overline{q}_2)\backslash\text{int } I^{\epsilon_1}$ converges to a point in $\mathcal{H}_{12}$ as $\tau \to -\infty$.

Now we will describe (locally) the stable manifold $W_s^\infty(\overline{q}_1, \overline{q}_2)$ which we define via

$$(8.32) \qquad W_s^\infty(\overline{q}_1, \overline{q}_2) = \{q(\tau) \text{ satisfying (8.4)} \mid q(\tau) \text{converges to an element of } \mathcal{H}_{12} \text{ as } \tau \to \infty\}.$$

As was the case for $W_u^\infty(\overline{q}_1, \overline{q}_2)$, we could have started out with solutions of (8.4) whose limit set, as $\tau \to \infty$, has a nonempty intersection with $\mathcal{H}_{12}$. However the analysis of critical points at infinity carried out in (8.6)-(8.21) shows that such trajectories converge to an element of $\mathcal{H}_{12}$. Thus (8.32) is an acceptable definition for $W_s^\infty(\overline{q}_1, \overline{q}_2)$. For any $q(\tau) \in W_s^\infty(\overline{q}_1, \overline{q}_2)$, there exists τ_0 such that

$$(8.33) \qquad q(\tau) \in \mathcal{V}_1 \quad \text{for} \quad \tau \geq \tau_0.$$

For such τ, $q(\tau)$ satisfies (8.5) which implies

$$(8.34) \qquad (q_1, q_2)(\tau) \in W_s(\overline{q}_1, \overline{q}_2) \quad \text{for} \quad \tau \geq \tau_0.$$

Let us consider a neighborhood Σ of $(\overline{q}_1, \overline{q}_2)$ in $W_s(\overline{q}_1, \overline{q}_2)$ having the following properties:

(i) Σ is invariant under the flow

$$\frac{d}{d\tau}(q_1, q_2) = -Z_{12}(q_1, q_2)$$

for $\tau \geq 0$.

(ii) For any $(q_1, q_2) \in \Sigma$,

$$\sum_{i=1}^{2} \|q_i - \frac{1}{2}[q_1 + q_2]\|_{L^\infty} < \frac{C_1}{2}.$$

Since by (4.15),

$$\sum_{i=1}^{2} \|\overline{q}_i - \frac{1}{2}[\overline{q}_1 + \overline{q}_2]\|_{L^\infty} \leq \frac{C_1}{4},$$

such a set Σ can be found. Clearly for any $q(\tau) \in W_s^\infty(\overline{q}_1, \overline{q}_2)$, there exists a $\tau_1 \geq 0$ such that

$$(8.35) \qquad (q_1, q_2)(\tau) \in \Sigma \quad \text{for} \quad \tau \geq \tau_1 \geq \tau_0.$$

Moreover since $\tau_1 \geq \tau_0$, we have

$$\frac{1}{2}\int_0^1 |\dot{Q}_3|^2(\tau)dt + \frac{1}{1+\|[Q_3 - \frac{1}{2}(q_1+q_2)]\|^2(\tau)} < \frac{\beta(C_1)}{2} \tag{8.36}$$

for $\tau > \tau_1$. Using (8.35)-(8.36), the analysis carried out in (8.6)-(8.21) holds and any orbit of (8.4) satisfying (8.35)-(8.36) lies in $W_s^\infty(\overline{q}_1, \overline{q}_2)$. Observe that if

$$(q_1, q_2)(0) \in \Sigma \tag{8.37}$$

and

$$\frac{1}{2}\int_0^1 |\dot{Q}_3|^2(0)dt + \frac{1}{1+\|[Q_3 - \frac{1}{2}(q_1+q_2)]\|^2(0)} < \frac{\beta(C_1)}{2}, \tag{8.38}$$

then by properties (i)-(ii) of Σ, if $q(\tau) = (q_1(\tau), q_2(\tau), Q_3(\tau))$ is the solution of (8.4) with initial data $q(0), q(\tau)$ satisfies (8.37)-(8.38) for any $\tau > 0$. Indeed (8.37)-(8.38) show (8.4) has the form (8.5) for small $\tau > 0$. Hence properties (i)-(ii) of Σ and the fact that the left hand side of (8.36) is nonincreasing with τ via (8.5) imply that (8.4) has the form (8.5) for all $\tau > 0$ and that (8.37)-(8.38) holds for any $\tau \geq 0$ with such initial data.

Now (8.33)-(8.34), (8.37)-(8.38) and our above remarks show that $W_s^\infty(\overline{q}_1, \overline{q}_2)$ can be described locally as a bundle over Σ with fiber $H_{(q_1,q_2)}$ where for $(q_1, q_2) \in \Sigma$,

$$H_{(q_1,q_2)} = \left\{ Q_3 \in W^{1,2} \mid \frac{1}{2}\int_0^1 |\dot{Q}_3|^2 dt + \frac{1}{1+\|[Q_3 - \frac{1}{2}(q_1+q_2)]\|^2} \leq \frac{\beta(C_1)}{2} \right\}. \tag{8.39}$$

Let

$$c = J_{12}(\overline{q}_1, \overline{q}_2). \tag{8.40}$$

We will make a particular choice of Σ. Let

$$\Sigma = \Sigma(\epsilon) = \{(q_1, q_2) \in W_s(\overline{q}_1, \overline{q}_2) \mid J_{12}(\overline{q}_1, \overline{q}_2) < c + \epsilon\} \tag{8.41}$$

Certainly (i) hold for small ϵ. Moreover since $(\overline{q}_1, \overline{q}_2)$ satisfies (4.15) and $\Sigma(\epsilon) \to (\overline{q}_1, \overline{q}_2)$ in $W^{1,2}$ as $\epsilon \to 0$, a fortiori $\Sigma(\epsilon) \to (\overline{q}_1, \overline{q}_2)$ in L^∞ and (ii) follows for $\epsilon \leq \epsilon_2$ where ϵ_2 depends on C_1 (and M). We can further assume that

$$\beta(C_1) < \epsilon_2(C_1) \tag{8.42}$$

for all $(\overline{q}_i, \overline{q}_j) \in \mathcal{K}_{ij}^{M+1}$. Let

$$0 < \frac{10\beta_1}{9} < \epsilon < \frac{\beta(C_1)}{2} \tag{8.43}$$

and set

$$W_s^\infty(\epsilon)(\overline{q}_1, \overline{q}_2) = \{q \in W_s^\infty(\overline{q}_1, \overline{q}_2) \mid \tilde{I}(q) < c + \epsilon\}. \tag{8.44}$$

Dropping the $(\overline{q}_1, \overline{q}_2)$ for convenience, $W_s^\infty(\epsilon)$ is clearly a connected neighborhood of $\mathcal{H}_{12}$ in $W_s^\infty(\overline{q}_1, \overline{q}_2)$. We will describe $W_s^\infty(\epsilon) \cap \overline{\mathcal{V}}_1$. On $\mathcal{V}_1$,

$$\tilde{I}(q) = J_{12}(q_1, q_2) + \frac{1}{2}\int_0^1 |\dot{Q}_3|^2 dt + \frac{1}{1 + \|[Q_3 - \frac{1}{2}(q_1 + q_2)]\|^2} \tag{8.45}$$

and the flow has the simple form (8.5).

Consider the following set:

$$\mathcal{C}_1 = \{q(0) \in W_s^\infty(\epsilon) \mid \text{the solution } q(\tau) \text{ of (8.5) remains in } \overline{\mathcal{V}}_1 \text{ for all}$$
$$\tau \geq 0 \text{ and } (q_1(\tau), q_2(\tau) \in \Sigma(\epsilon)\}.$$

The argument used above in (8.34)-(8.39) shows that if $q(0) \in W_s^\infty(\epsilon)$, there is a $\tau_0 > 0$ such that $q(\tau) \in \mathcal{C}_1$ for $\tau \geq \tau_0$. We claim that in fact $q(\tau) \in \mathcal{C}_1$ for all $\tau \geq 0$, i.e.

$$W_s^\infty(\epsilon) = \mathcal{C}_1. \tag{8.46}$$

Indeed assume that $q(\tau_1) \in \partial\mathcal{C}_1$ and $q(\tau) \in \mathcal{C}_1$ for $\tau \geq \tau_1$, $\tau_1 > 0$. Since (8.5) holds on $[\tau_1, \infty)$, $(q_1(\tau_1), \ q_2(\tau_1)) \in W_s(\overline{q}_1, \overline{q}_2)$. Since (8.45) holds and since $q(\tau_1) \in$ int $\tilde{I}_{c+\epsilon}$ $(\tau_1 > 0)$, $(q_1(\tau_1), q_2(\tau_1)) \in W_s(\overline{q}_1, \overline{q}_2) \cap$ Int $\tilde{J}_{12,c+\epsilon'}$ and hence $(q_1(\tau_1), q_2(\tau_1)) \in$ int $\Sigma(\epsilon)$. Therefore $q(\tau_1) \in \partial\mathcal{V}_1$ (since $q(\tau_1) \in \partial\mathcal{C}_1$ and $(q_1(\tau_1), q_2(\tau_1)) \in$ int $\Sigma(\epsilon)$ and hence $(q_1(\tau_1), q_2(\tau_1)) \notin \partial\Sigma(\epsilon)$. This implies that either

$$\sum_{i=1}^{2} \|q_i - \frac{1}{2}[q_1 + q_2]\|_{L^\infty} = \frac{C_1}{2}$$

which is impossible by (ii) of the definition of $\Sigma(\epsilon)$ or that

$$\Psi(Q_3) = \frac{\beta(C_1)}{2} > \epsilon$$

which is also impossible since then

$$\tilde{I}(q(\tau_1)) \geq c + \frac{\beta(C_1)}{2} > c - \epsilon, \tag{8.47}$$

a contradiction. Hence (8.46) holds.

Using (8.45) and the definition of C_1, $W_s^\infty(\epsilon)$ is a bundle over $\Sigma(\epsilon)$ with a fiber as described in 3°b of Theorem 8.2. Moreover $W_s^\infty(\epsilon)$ is a Fredholm manifold. Indeed $\Sigma(\epsilon)$ is a Fredholm manifold since J_{12}' is Fredholm and proper near $(\overline{q}_1, \overline{q}_2)$. Thus Z_{12} may be chosen to be Fredholm and proper near $(\overline{q}_1, \overline{q}_2)$. The set $G_{(q_1,q_2)}$ is also a Fredholm manifold and $W_s^\infty(\epsilon)$ inherits this from the product structure. This completes the proof of 3° (and 4°) of Theorem 8.2.

To obtain 5° and 6°, a few preliminaries are needed. Note first that if

$$\frac{10\beta_1}{9} < \epsilon < \frac{\beta(C_1)}{2} \tag{8.48}$$

then $W_s^\infty(\epsilon)$ is a uniform neighborhood of $\mathcal{H}_{12}$ in $W_s^\infty(\overline{q}_1, \overline{q}_2)$. Indeed if $q \in \mathcal{H}_{12}$,

$$\tilde{I}(q) = J_{12}(\overline{q}_1, \overline{q}_2) + \frac{1}{1 + \|[Q_3 - \frac{1}{2}(\overline{q}_1 + \overline{q}_2)]\|^2} \leq c + \beta_1 < c + \epsilon. \tag{8.49}$$

Observe also that (8.5) holds on $W_s^\infty(\epsilon)$ and provides us with the following information about the behavior of the (decreasing) flow restricted to $W_s^\infty(\epsilon)$:

$$\begin{cases} \frac{d}{d\tau}(q_1, q_2) = -Z_{12}(q_1, q_2) & (q_1, q_2) \in \Sigma(\epsilon) \\ \dot{Q}_3(\tau) = e^{-\tau}\dot{Q}_3(0) \\ w(\tau) = [Q_3 - \frac{1}{2}(q_1 + q_2)](\tau) = \lambda(\tau)w(0) \end{cases} \tag{8.50}$$

where $\lambda(\tau)$ converges to a limit ν as $\tau \to \infty$ such that $|\nu| \geq (1/\beta_1)^{-1} - 1)^{1/2}$. Lastly observe that a Morse Lemma is available around $\mathcal{H}_{12}$ for $\tilde{Z}$ in the following sense: Let $\mathcal{N}_1 \subset \overline{\mathcal{N}}_1 \subset \operatorname{int} \mathcal{N}_2 \subset \mathcal{N}_2$ be neighborhoods of $(\overline{q}_1, \overline{q}_2)$ in Λ_{12} such that

$$Z_{12}(X, Y) = (-X, Y) \tag{8.51}$$

in $\mathcal{N}_2$ where (X,Y) are coordinates along $W_u(\overline{q}_1,\overline{q}_2)$ and $W_s(\overline{q}_1,\overline{q}_2)$ respectively and

$$\sum_{j=1}^{2}\|q_j-\frac{1}{2}[q_1+q_2]\|_{L^\infty}<\frac{C_1}{2} \tag{8.52}$$

for all $(q_1,q_2)\in\mathcal{N}_i,\ \ i=1,2$.

For $i=1,2$, let

$$\mathcal{W}_i=\left\{(q_1,q_2,Q_3)\mid(q_1,q_2)\in\mathcal{N}_i\quad\text{and}\quad\tilde{\omega}_{12}(q_1,q_2,Q_3)\geq\frac{1}{2i}\right\}. \tag{8.53}$$

Then by the definition of $\tilde{\omega}_{12}$, $\mathcal{W}_i\subset\mathcal{V}_1$. Moreover since $\tilde{\omega}_{12}(q)<\frac{1}{2}$ on $\mathcal{W}_2\backslash\mathcal{W}_1$, by the argument of Lemma 4.14, there are constants γ and K_1 such that

$$\begin{cases}\tilde{I}'(q)\tilde{Z}(q)\geq\gamma>0\\[2ex] \|\tilde{Z}(q)\|_{W^{1,2}}\leq K_1\end{cases} \tag{8.54}$$

for all $q\in\mathcal{W}_2\backslash\mathcal{W}_1$. Furthermore, (8.5) holds in $\mathcal{W}_2$ and shows that if $q(\tau)$ is a solution of (8.5) such that $q(\tau)\in\mathcal{W}_2$ for $\tau\in[0,\tau_0]$, then $q(\tau)=(X,Y,Q_3)(\tau)$ with

$$\begin{cases}X(\tau)=e^{\tau}X(0)\\[2ex] Y(\tau)=e^{-\tau}Y(0)\\[2ex] \dot{Q}_3(\tau)=e^{-\tau}\dot{Q}_3(0)\\[2ex] w(\tau)\equiv[Q_3-\frac{1}{2}(q_1+q_2)](\tau)=\lambda(q(\tau))w(0)\end{cases} \tag{8.55}$$

where

$$\frac{d}{d\tau}\lambda(q(\tau))\geq0\quad\text{and}\quad\rightarrow0$$

if and only if dist $(q(\tau),\mathcal{V}_0)\rightarrow0$. Now (8.54)-(8.55) show that $\mathcal{W}_2$ has similar properties to neighborhoods of critical points for functionals satisfying the Palais-Smale condition. Indeed the explicit formulas of (8.55) show (PS) is satisfied in $\mathcal{W}_2$ along any given trajectory. Moreover $\mathcal{W}_2$ is a neighborhood of $\mathcal{H}_{12}$ on which $\tilde{Z}$ has the reduction provided in (8.55) which splits along the (q_1,q_2,Q_3) coordinates yielding a product structure for the flow corresponding to $-\tilde{Z}$.

As a last preliminary observe that if $\mathcal{N}_2$ is small enough, $\mathcal{N}_2 \subset \mathcal{N}(\rho)$ as defined following Proposition 4.2 and any solution $(q_1, q_2)(\tau)$ of (8.5)(i) starting in $\mathcal{N}_2$ satisfies (8.52) for any τ such that $J_{12}(q_1(\tau), q_2(\tau)) \geq \frac{\epsilon_1}{2}$. This implies that (8.5) holds for any decreasing flow trajectory $q(\tau)$ starting in $\mathcal{W}_2$ as long as $\tilde{I}(q(\tau)) \geq \epsilon_1$ (and therefore $J_{12}(q_1(\tau), q_2(\tau)) \geq \frac{\epsilon_1}{2}$ since $\beta(C_1) < \epsilon_1$; see (8.5)-(8.7)).

Now we are ready for the proofs of $5° - 6°$. These proofs are essentially the same as those of 2° and 5° of Theorem 7.2. However there are a few differences which will be indicated next. The proof of 2° of Theorem 7.2 relied on a two step induction. In particular, recalling the idea of the proof, given two consecutive critical values $c_1 < c_2$ with corresponding critical points x_1, x_2, $W_s(x_1)$ and $W_u(x_2)$ intersect transversally, strongly and uniformly. This insures that if $c_0 < c_1$ and $W_s(x_0)$ intersects $W_u(x_1)$ transversally, then $W_u(x_2)$ intersects $W_s(x_0)$ transversally in a neighborhood of $W_u(x_1)$. Therefore if we want to guarantee that $W_u(x_2)$ intersects $W_s(x_0)$ transversally, we need only take care of a part of $W_u(x_2)$ which does not lie in a neighborhood of $W_u(x_1)$. Since we are interested in $W_u(x_2) \cap W_s(x_0)$, we may consider $W_u(x_2) \cap f^{-1}(c)$ for $c \in (c_0, c_1)$. Then the part of $W_u(x_2) \cap f^{-1}(c)$ which does not lie in the given neighborhood of $W_u(x_1)$ is compact and making it transversal to $W_s(x_0)$ follows from the standard transversality theorem [16].

As in the proof of 2° of Theorem 7.2, in order to insure the transversal intersection in the strong and uniform sense of $W_s(x_1)$ and $W_u(x_2)$, or $W_s(x_0)$ and $W_u(x_1)$, (PS) is needed outside of suitable neighborhoods $\mathcal{O}_i$ of the critical set including the critical points at infinity. These neighborhoods should be small enough so that they do not intersect a level set $f^{-1}(c)$ for a fixed c between two critical levels. In order to guarantee that the strong and uniform transversal intersection of $W_s(x_0)$ and $W_u(x_1)$ implies that of $W_s(x_0)$ and $W_u(x_2)$ in a suitable neighborhood of $W_u(x_1)$, a Morse Lemma is needed for the flow in the $\mathcal{O}_i$'s. Since (PS) is satisfied outside the $\mathcal{O}_i$'s, the part of $W_u(x_2)$ which is then left intersects $f^{-1}(c)$, for $c \in (c_0, c_1)$, in a compact set. Thus our induction can continue. Neighborhoods $\mathcal{O}_i$ are available for our present framework, i.e. for $\tilde{I}$ between the levels ϵ_1

and $M+1$, at least to the extent that (PS) is satisfied outside of these sets. The sets $\mathcal{W}_2$ defined in (8.51)-(8.55) can be used as the $\mathcal{O}_i$'s for critical points at infinity. The fact that the flow $\eta(\tau,\cdot)$ is Fredholm and locally proper near the remaining critical points in $\tilde{I}^{-1}(\epsilon_1, M+1)$ provides $\mathcal{O}_i$'s for such points.

Now we will be more precise. Let $c, c' \in [\epsilon_1, M+1]$ be noncritical values of $\tilde{I}$. The critical points at infinity, i.e. sets of type $\mathcal{H}_{12}$ provide us with a continuum of critical values for $\tilde{Z}$. Namely corresponding to $(\overline{q}_i, \overline{q}_j)$, we have $J_{ij}(\overline{q}_i, \overline{q}_j) + \epsilon$ for any $\epsilon \in [0, \beta_1]$. This is, of course, an artiface of the method we are using which introduces a vector field, $\tilde{Z}$, with a hyperbolic manifold of rest points at each level where (PS) fails. Nevertheless we will argue as if this manifold were a single point. The hyperbolic structure displayed in $1° - 2°$ of Theorem 8.2, in (8.31) and (8.55) allows us to do so. For the sake of precision note that a noncritical level c will either satisfy $c > J_{ij}(\overline{q}_i, \overline{q}_j) + \beta_1$ or $c < J_{ij}(\overline{q}_i, \overline{q}_j)$. Since β_1 satisfies (8.1), these critical interval levels do not overlap.

For any classical critical point, $\overline{q}$, of $\tilde{I}$ and in particular for those in $\tilde{I}^{-1}(\epsilon_1, M+1)$, $W_u(\overline{q})$ is finite dimensional. Therefore if $c' < \tilde{I}(\overline{q})$ and is larger than the next critical level of $\tilde{I}$, $W_u(\overline{q}) \cap \tilde{I}^{-1}(c')$ is compact. Since (PS) holds outside $\{\mathcal{O}_i\}$, the first step of the induction argument of 2° of Theorem 7.2 is possible for $\overline{q}$. This ensures a transversal intersection of $W_u(\overline{q})$ with $W_s(\overline{q}')$ or $W_s^\infty(\overline{q}')$ at the next critical level since in both cases (W_s or W_s^∞), we will have to ensure the uniform and strong transversal intersection of a compact manifold with a Fredholm (possibly unbounded) manifold. Given another critical level $c'' < \tilde{I}(\overline{q})$ which corresponds to a classical critical point or to a critical point at infinity, the same argument guarantees the strong and uniform transversal intersection of $W_s(\overline{q}'')$ or $W_s^\infty(\overline{q}'')$ with the trajectories originating in $W_u(\overline{q}) \cap \tilde{I}^{-1}(c')$ which do not enter the $\mathcal{O}_i$'s between the levels c and c'. Here $c'' < c$ and c is less than the next larger critical level. Thus the first step of the induction argument of 2° of Theorem 7.2 is available for classical critical points. The second step is also, since it relies on a Morse reduction about a classical critical point. This is available here by the local properness and Fredholm

character of $\tilde{I}'$ and $\tilde{Z}$.

Thus we are left with critical points at infinity. For such points the following direct argument shows that both steps of the induction procedure are available. First observe that for any $(\overline{q}_i, \overline{q}_j)$ and $\overline{q}'$ such that

$$\epsilon_1 \le \tilde{I}(\overline{q}') < J_{ij}(\overline{q}_i, \overline{q}_j) \le M + 1,$$

$$W_u^\infty(\overline{q}_i, \overline{q}_j) \cap W_s(\overline{q}') = \phi. \tag{8.56}$$

Indeed (8.31) shows that the decreasing flow, restricted to $W_u^\infty(\overline{q}_i, \overline{q}_j)$ splits in the product structure between $W_u(\overline{q}_i, \overline{q}_j)$ and the fiber with $[Q_r - \frac{1}{2}(q_i, q_j)](\tau) \equiv$ constant $= \gamma$. Moreover $|\gamma| \ge (1/\beta_1)^{-1} - 1)^{1/2}$. If β_1 is small enough, this fact implies (8.56) for then if $\overline{q}'$ has the form $(\overline{q}'_i, \overline{q}'_j, \overline{Q}'_r)$,

$$\left(\frac{1}{\beta_1} - 1\right)^{1/2} > |[\overline{Q}_r - \frac{1}{2}(\overline{q}'_i + \overline{q}'_j)]| \tag{8.57}$$

while if $\overline{q}'$ cannot be so represented, no constraint is needed for β_1.

The remaining case to consider is $W_u^\infty(\overline{q}_i, \overline{q}_j) \cap W_s^\infty(\overline{q}'_i, \overline{q}'_j)$ where the indices i, j are the same for these sets since the sets $\overline{\mathcal{V}}_1(i,j)$ which contain $W_u^\infty(\overline{q}_i, \overline{q}_j)$ are pairwise disjoint. Using (8.31) again shows $W_u^\infty(\overline{q}_i, \overline{q}_j) \cap W_s^\infty(\overline{q}'_i, \overline{q}'_j)$ is a bundle over $W_u(\overline{q}_i, \overline{q}_j) \cap W_s(\overline{q}'_i, \overline{q}'_j)$ described via

$$\begin{aligned} W_u(\overline{q}_i, \overline{q}_j) \cap W_s^\infty(\overline{q}'_i, \overline{q}'_j) = \{(q_i, q_j, Q_r) \mid (q_i, q_j) \in W_u(\overline{q}_i, \overline{q}_j) \cap W_s(\overline{q}'_i, \overline{q}'_j) \\ \text{and } |[Q_r - \frac{1}{2}(q_i + q_j)]| \ge \left(\frac{1}{\beta_1} - 1\right)^{1/2} \}. \end{aligned} \tag{8.58}$$

Since $W_u(\overline{q}_i, \overline{q}_j)$ and $W_s(\overline{q}'_i, \overline{q}'_j)$ are assumed inductively to intersect transversally, strongly and uniformly, $W_u^\infty(\overline{q}_i, \overline{q}_j)$ and $W_s^\infty(\overline{q}'_i, \overline{q}'_j)$ also do so. Observe that the transversality occurs in the base of the bundles W_u^∞ and W_s^∞, not in the $[Q_r]$ fibers. Now recall that we chose $\mathcal{N}_2 \subset \mathcal{N}(\rho)$ so that any decreasing flow trajectory starting in a set $\mathcal{W}_2$ satisfies (8.5) as long as $\tilde{I}(q(\tau)) \ge \epsilon_1$. This fact, together with (8.55) and the transversality in the base

of the bundles which was just pointed out implies that the second inductive step is also available for critical points at infinity: if $W_u(\overline{q})$ or $W_u^{\infty}(\overline{q})$ intersects $W_s^{\infty}(\overline{q}')$ transversally, strongly and uniformly, between the levels ϵ_1 and $M+1$, then $W_u(\overline{q})$ or $W_u^{\infty}(\overline{q})$ will intersect any other $W_s^{\infty}(\overline{q}'')$ transversally, strongly and uniformly in a neighborhood of $W_u^{\infty}(\overline{q}')$ provided that $W_u^{\infty}(\overline{q}')$ intersects $W_s^{\infty}(\overline{q}'')$ transversally, strongly and uniformly. Thus we have 5° of Theorem 8.2.

Now we turn to 6° of Theorem 8.2. It is almost simpler to prove it here than in Theorem 7.2 due to the representation we have for the flow at infinity, in particular (8.31). However a complication is created due to that fact that 3° of Theorem 7.2 does not hold here. Property 3° was used in both the ENR and retraction parts of 5° of Theorem 7.2 so we must study this situation carefully.

To see what happens to 3°, we consider a simple case where $q \in \mathcal{K}(\tilde{I})$ and the largest critical value c smaller than $\tilde{I}(q)$ occurs at ∞, i.e. $\tilde{Z}$ has a set of rest points, say $\mathcal{H}_{12}(\overline{q}_1, \overline{q}_2)$ with stable manifold $W_s^{\infty}(\overline{q}_1, \overline{q}_2)$ and $J_{12}(\overline{q}_1, \overline{q}_2) = c$. Let $\frac{10}{9}\beta_1 < \epsilon < \frac{\beta(C_1)}{2}$ and consider $W_u(q) \cap W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap \tilde{I}^{-1}(c+\epsilon)$. Our assumptions on ϵ, $\tilde{I}(q), c$ imply that $\mathcal{H}_{12}(\overline{q}_1, \overline{q}_2) \subset I^{c+\frac{2}{10}\epsilon}$. Moreover $W_u(q)$ being finite dimensional, $W_u(q) \cap I^{-1}(c+\epsilon)$ is homeomorphic to a finite dimensional sphere and therefore is compact. Consequently $W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap W_u(q) \cap \tilde{I}^{-1}(c+\epsilon)$ is compact. It is also useful to observe at this point that the intersection $W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap W_u(q) \cap \tilde{I}^{-1}(c+\epsilon)$ can be made transversal in a standard way since it represents the intersection of a compact manifold with a Fredholm, and closed manifold.

Since $\epsilon < \frac{\beta(C_1)}{2}$, the description of $W_s^{\infty}(\overline{q}_1, \overline{q}_2)$ given in 3°b of this theorem, shows $W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap W_u(q) \cap \tilde{I}^{-1}(c+\epsilon) \subset \mathcal{V}_1(1,2)$. With C_1 sufficiently lage and $\beta(C_1)$ small, it may be assumed that if $(q_1, q_2, Q_3) \in W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap \tilde{I}^{c+\epsilon}$, then $(q_1, q_2) \in \Sigma$ as defined after (8.34) and Q_3 satisfies (8.36). Then as noted earlier, the flow for $-\tilde{Z}$ on $W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap \tilde{I}^{c+\epsilon}$ takes the form (8.5), i.e. splits nicely. In particular this form holds on $W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap W_u(q) \cap \tilde{I}^{-1}(c+\epsilon)$. Therefore $W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap W_u(q) \cap \tilde{I}^{c+\epsilon}$ is the image under this flow of the compact set $W_s^{\infty}(\overline{q}_1, \overline{q}_2) \cap W_u(q) \cap \tilde{I}^{-1}(c+\epsilon)$.

It was shown earlier in (8.12)-(8.13) that for a trajectory in $W_s^\infty(\overline{q}_1, \overline{q}_2)$, the limit, μ, of $(1 + |[Q_3 - \frac{1}{2}(q_1 + q_2)]|^2(\tau))^{-1}$ as $\tau \to \infty$ is nonzero. Thus $|[Q_3 - \frac{1}{2}(q_1 + q_2)]|(\tau)$ does not tend to ∞ along this trajectory. The same is true uniformly for all trajectories originating in the compact set $W_s^\infty(\overline{q}_1, \overline{q}_2) \cap W_u(q) \cap \tilde{I}^{-1}(c+\epsilon)$. Hence 3° of Theorem 7.2 does not hold here. Only part of $\mathcal{H}_{12}$ lies in $\overline{W}_u(q)$ due to the fact that the single critical point occurring in Theorem 7.2 is replaced by the entire set, $\mathcal{H}_{12}$, here. However as the above remarks show, we do have transversal intersections at least for the critical points at infinity and also for the intersection of the unstable manifold of a classical critical point with the stable manifold of a critical point at infinity if there is no other critical value between the two critical values considered.

In the classical framework, as in the proof of 3° of Theorem 7.2 and 5° of Theorem 8.2, these facts together with the Morse Lemma allowed us to conclude that all intersections, without restriction, were transversal. The retraction result then followed. In the present situation, the Morse Lemma available for the critical points at infinity is special since we have manifolds of critical points at infinity. We could break up these manifolds into a finite number of points by modifying the functional, and then use the finite dimensional result. However, this leads to new technical problems so we prefer to argue directly. The problem is the following: since we have a whole set $\mathcal{H}_{12}$ of critical points at infinity, the sum of the tangent space to $W_u^\infty(\overline{q}_1, \overline{q}_2)$ and the tangent space to $W_s^\infty(\overline{q}_1, \overline{q}_2)$ at any of these critical points is not direct. There is a direct sum: the sum of the tangent space to $W_s^\infty(\overline{q}_1, \overline{q}_2)$ at $\overline{q} = (\overline{q}_1, \overline{q}_2, \overline{Q}_3)$ and the subspace of the tangent space at $\overline{q}$ defined in the coordinates $(q_1, q_2, Q_3 - [Q_3],\ [Q_3 - \frac{q_1+q_2}{2}])$ by

$$T_{(\overline{q}_1, \overline{q}_2)} W_u(\overline{q}_1, \overline{q}_2) \times \{0\}. \tag{8.59}$$

In (8.59), $T_{(\overline{q}_1, \overline{q}_2)} W_u(\overline{q}_1, \overline{q}_2)$ refers to the tangent space at $(\overline{q}_1, \overline{q}_2)$ to $W_u(\overline{q}_1, \overline{q}_2)$; $\{0\}$ refers to the zero in the tangent space to $W_T^{1,2}([0,T], \mathbf{R}^\ell)$ with coordinates $(Q_3 - [Q_3],$ $[Q_3 - \frac{1}{2}(q_1 + q_2)])$.

Comparing (8.59) with (7.25), we should take E^- here at $(q_1, q_2) \in W_u(\overline{q}_1, \overline{q}_2)$, to be $T_{(q_1,q_2)}W_u(\overline{q}_1, \overline{q}_2) \times \{0\}$ (using the same coordinates). With the above definition of E^-, E^- is invariant under the linearized flow (see (8.31)). With this modification, the statement of Proposition 7.31 holds. After (7.38), we established (i), (ii) and (iii) inductively. Here, due to the presence of the critical points at infinity, (ii) cannot hold and is replaced by the inclusion:

(ii)′ $\overline{W_u}(q) \cap f_c \subset$

$$\subset W_u(q) \cup \{W_u(q') \cap f_c \mid q' \in F_q\} \cup \{W_u^\infty(\overline{q}_i, \overline{q}_j) \cap f_c \mid W_u(q) \cap W_s^\infty(\overline{q}_i, \overline{q}_j) \neq \phi\}$$

(iii) remains the same (when generalized in order to take into account the critical points at infinity). With this modification, (i), (ii)′ and (iii) hold and their proof is nearly the same as in the classical case. The modifications are related to the fact that (7.24) does not hold here; i.e. there is no local fibration of $W_u(q)$ over $W_u^\infty(\overline{q}_1, \overline{q}_2)$ if $W_u(q) \cap W_s^\infty(\overline{q}_1, \overline{q}_2) \neq \phi$, due to the fact that $\overline{W}_u(q)$ does not necessarily contain all of $W_u^\infty(\overline{q}_1, \overline{q}_2)$ in its closure. However, the arguments using the fibration may be replaced, for the proof of (i), (ii)′ and (iii), by the transversality, property, i.e. by Proposition 7.31.

Again using the above definition of E^-, Proposition 7.64 holds in this extended framework, the proof being essentially the same. This proposition provides us with a neighborhood of $I^{\epsilon_1} \cup \mathcal{D}_{M+1}^\infty \cup \mathcal{D}_{M+1}$, invariant under the decreasing flow, of the type $\mathcal{W}^\infty \cup \mathcal{V}$, where $\mathcal{V}$ is a neighborhood of $\mathcal{D}_{M+1}$ with a piecewise smooth boundary, intersecting $\mathcal{W}^\infty$ transversally. $\mathcal{W}^\infty$ also has a piecewise smooth boundary. $\mathcal{W}^\infty$, $\mathcal{V}$ and $\mathcal{V} \cap \mathcal{W}^\infty$ are therefore ANR's.

For later purposes, we point out that we may suppose that $-\tilde{Z}$ points inwards on $\mathcal{W}^\infty \cup \mathcal{V}$. This is obtained by modifying slightly the deformation argument of (7.66) - (7.72). We have shown there, up to a change in notation, that we can find a vector field v, which vanishes on $I^{\epsilon_1} \cup \mathcal{D}_{M+1}^\infty \cup \mathcal{D}_{M+1}$ and such that v points inwards on $\mathcal{W}^\infty \cup \mathcal{V}$. We were using v in order to show that $-\tilde{Z}(q) + \epsilon v$ points inwards on $\mathcal{W}^\infty \cup \mathcal{V}$. We can argue differently and assume that $\mathcal{W}^\infty \cup \mathcal{V}$ is constructed by using the flow of $-\tilde{Z}(q) + \epsilon v$

in (7.58) (instead of the flow of $-\tilde{Z}(q)$), with ϵ sufficiently small. If $-\tilde{Z}(q)$ is tangent at any point q to such a $\mathcal{W}^\infty \cup \mathcal{V}$, then v has to point outwards to $\mathcal{W}^\infty \cup \mathcal{V}$ at such a q. However, when $\epsilon \to 0$ the boundary of $\mathcal{W}^\infty \cup \mathcal{V}$ approaches the boundary of the similar set for $-\tilde{Z}(q)$ and v points inwards on this set. Therefore, we have a contradiction, and $-\tilde{Z}(q)$ is transverse to $\mathcal{W}^\infty \cup \mathcal{V}$ along its boundary.

Using the same kind of argument as in §7, to establish that $W_u(a,b)$ is an ENR, we can prove that $\mathcal{D}_{M+1}$ is an ENR. Namely, $\mathcal{D}_{M+1}$ is locally contractible for the same reason $W_u(a,b)$ is locally contractible. $\mathcal{D}_{M+1}$ is locally compact since $\mathcal{D}_{M+1}$ is a union of finite dimensional manifolds and since $\mathcal{D}_{M+1}$ is locally closed. (Indeed $\overline{\mathcal{D}_{M+1}} \subset \mathcal{D}_{M+1} \cup \mathcal{D}^\infty_{M+1} \cup I^{\epsilon_1}$). Since $-\tilde{Z}$ is tangent to $\mathcal{D}_{M+1}$ and on $\partial\mathcal{W}^\infty$, it points inwards to $\mathcal{W}^\infty$, $\mathcal{W}^\infty \cap \mathcal{D}_{M+1}$ is a retract of an open subset in $\mathcal{D}_{M+1}$ (namely $\bigcup_{s\in\mathbb{R}} \eta(s, \mathcal{D}_{M+1} \cap \mathcal{W}^\infty)$) and therefore is also an ENR of dimension at most m. Thus we have established that $\mathcal{V}_\epsilon, W^\infty, W^\infty \cap \mathcal{V}_\epsilon$ are ANR's and that $\mathcal{D}_{M+1}$ and $\mathcal{D}_{M+1} \cap \mathcal{W}^\infty$ are ENR's of dimension at most m.

The sets $\mathcal{V}_\epsilon$, $\mathcal{W}^\infty$, depend on small positive parameters $\epsilon_1, \dots, \epsilon_r$ (respectively $\epsilon_1^\infty, \dots, \epsilon_k^\infty$). These parameters allow us to define the balls $B(x_i, \epsilon_i)$ in the proof of Proposition 7.64, from which the set $\mathcal{V}_\epsilon$ (and $\mathcal{W}^\infty$) is constructed. These ϵ_i and ϵ_i^∞ obey constraints of the type:

$$0 < \epsilon_i < \varphi_i(\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_1^\infty, \dots, \epsilon_{i-1}^\infty);$$

$$0 < \epsilon_i^\infty < \psi_i(\epsilon_1, \dots, \epsilon_{i-1}, \epsilon_1^\infty, \dots, \epsilon_{i-1}^\infty).$$

Since $W_u^\infty(\overline{q}_r, \overline{q}_\ell) \cap W_s(\overline{q}) = \phi$, ψ_i does not depend on $\epsilon_1, \dots, \epsilon_{i-1}$, i.e. the constraints on ϵ_i^∞ are of the type

$$0 < \epsilon_i^\infty < \psi_i(\epsilon_1^\infty, \dots, \epsilon_{i-1}^\infty).$$

We may therefore fix $\epsilon_1^\infty, \dots, \epsilon_k^\infty$, thus obtaining the set $\mathcal{W}^\infty$ and consider $\mathcal{V}_\epsilon = \mathcal{V}(\epsilon_1, \dots, \epsilon_r)$ with small ϵ_j's. The intersection $\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty$ is transverse and is a piecewise smooth manifold. If $\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty$ were a manifold, then when $\epsilon_j \to 0$, $j = 1, \dots, r$, the pair $(\mathcal{V}(\epsilon), \mathcal{V}(\epsilon) \cap \mathcal{W}^\infty)$ would deform through an isotopy and we would then very easily obtain a retraction by deformation onto $(\mathcal{D}_{M+1}, \mathcal{D}_{M+1} \cap \mathcal{W}^\infty)$. The statement about the

homologies of $\mathcal{V}(\epsilon)$ and $\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty$ of Theorem 8.2 would follow.

Unfortunately, $\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty$ is only a piecewise manifold and therefore the retraction by deformation argument is trickier (see Bahri [15]). Observe however that if $0 < \epsilon_j' < \epsilon_j$, ($\epsilon_j$ small enough), then $\mathcal{V}(\epsilon')$ is a retract by deformation of $\mathcal{V}(\epsilon)$ and, using the fact that the intersection is transversal, $\mathcal{V}(\epsilon') \cap \mathcal{W}^\infty$ is a retract by deformation of $\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty$. Therefore, the homology of $\mathcal{V}(\epsilon)$ and the homology of $\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty$ do not depend on ϵ, for ϵ small. Since $\mathcal{D}_{M+1}$ and $\mathcal{D}_{M+1} \cap \mathcal{W}^\infty$ are ENR's and since $\bigcap_\epsilon \mathcal{V}(\epsilon) = \mathcal{D}_{M+1}$ and $\bigcap_\epsilon (\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty) = \mathcal{D}_{M+1} \cap \mathcal{W}^\infty$, $\mathcal{V}(\epsilon)$ has the homology of $\mathcal{D}_{M+1}$ and $\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty$ has the homology of $\mathcal{D}_{M+1} \cap \mathcal{W}^\infty$. (If we want to avoid the construction of the retractions by deformation of $\mathcal{V}(\epsilon)$ on $\mathcal{V}(\epsilon')$ and $\mathcal{V}(\epsilon) \cap \mathcal{W}^\infty$ on $\mathcal{V}(\epsilon') \cap \mathcal{W}^\infty$, we can consider Cech homology. The argument in (5.4) of section 5 then holds in Cech homology. Since $\mathcal{W}^\infty$ is an ANR, Cech homology coincides with the usual homology for $\mathcal{W}^\infty$; the same result holds for the other sets. Therefore, the argument may be continued as stated.)

The proof of Theorem 8.2 concludes now by showing that $\mathcal{W}^\infty$ has the homotopy type of $I^{\epsilon_1} \cup \mathcal{D}_{M+1}^\infty$. We observe that (8.5) holds in a neighborhood of $\mathcal{D}_{M+1}^\infty$ and we have the nice splitting situation already described. Therefore, in such a neighborhood ($\mathcal{V}_1$ for example), we may construct $\mathcal{W}^\infty$ out of a similar kind of neighborhood for the associated two-body problem and a neighborhood in $W^{1,2}$ of the set

$$\left\{ w \in \mathbf{R}^\ell \mid \frac{1}{1+|w|^2} \leq \beta_1 \right\}.$$

(Here w will be $Q_3 - [\frac{q_1+q_2}{2}]$).

Since the associated two-body problems satisfy the Palais-Smale condition and since the gradient of I_{ij} is Fredholm, the results of §7, in particular Theorem 7.2, generalize immediately to this framework. The neighborhoods considered provided by Proposition 7.64, have the homotopy type of the union of the unstable manifolds for the critical points of the two-body problems. This yields the result about $\mathcal{W}^\infty$, except for some minor details which we omit for simplicity.

§9. A refined version of Theorem 1.

In this section we will prove a refined version of Theorem 1 under further assumptions that the critical points of I are "nondegenerate". More precisely, let q be a critical point of I and let $m(q)$ denote its Morse index, i.e. $m(q)$ is the number of negative eigenvalues of $I''(q)$. Let $\overline{m}(q)$ denote the generalized Morse index of q, i.e. $\overline{m}(q) = m(q)+$ the number of 0 eigenvalues of $I''(q)$. By (2.24), $\overline{m}(q) - m(q) \geq \ell$. Observe also that the degeneracy directions, at a critical point, satisfy a second order ODE in $(\mathbf{R}^\ell)^3$. Therefore $\overline{m}(q) \leq m(q) + 6\ell$.

Let $\beta_k(\Lambda)$ be the k^{th} Betti number of Λ and let N_k denote the number of critical points, q, of I such that $m(q) = k$. Then we have

Theorem 3: Let V satisfy $(V_1) - (V_6)$. Assume that if $I'(q) = 0$ and $m(q) \leq k$ or $\overline{m}(q) \geq k$, then $\overline{m}(q) - m(q) = \ell$, i.e. q is a nondegenerate critical point of I modulo translations. Then

$$N_k \geq \beta_k(\Lambda) - 12 \quad \text{if} \quad k \geq 3\ell + 1. \tag{9.1}$$

Proof. The inequalities (9.1) can be interpreted as a version of the Morse inequalities. However, due to the fact that critical points of the two-body functionals I_{ij}, provide us with critical points at infinity and since we have no control on the number of these critical points, the standard proof of the Morse inequalities cannot be used here. The proof given here bypasses these difficulties (and also provides a proof of the Morse inequalities in the standard setting).

There exists $M > 0$ such that any homology class $[c]$ in $H_k(\Lambda)$ may be represented by a chain c having support in I^M. This is the case since $H_k(\Lambda)$ is finitely generated (Λ is the loop space of the fibration $p: Y_3 \to Y_2$, see section 5, with fiber equal to the wedge of two spheres $S^{\ell-1}$. That $H_k(\Lambda)$ is finitely generated follows from [17]). Let

$$\mathcal{K}_k^M = \{q \in \Lambda \mid I'(q) = 0, I(q) \leq M, \quad \text{and} \quad m(q) = k\}. \tag{9.2}$$

As was pointed out in §4, after a perturbation argument, it may be assumed that all critical points of I in I^M are nondegenerate (modulo translations). Set

$$\mathcal{K}^{k-1,M} = \{q \in \Lambda \mid I'(q) = 0, I(q) \leq M, \quad \text{and} \quad m(q) \leq k-1\}. \tag{9.3}$$

Let

$$A_k = \bigcup_{q \in \mathcal{K}_k^M} W_u(q) \tag{9.4}$$

and

$$\mathcal{B}_{k-1} = \bigcup_{q \in \mathcal{K}^{k-1,M}} W_u(q). \tag{9.5}$$

Using 6° of Theorem 8.2, the chain c with support in I^M representing $[c] \in H_k(\Lambda)$ may be represented as a chain in

$$H_k(\mathcal{W}^\infty \cup \mathcal{V}_\epsilon(\mathcal{D}_{M+1}))$$

with

$$\mathcal{D}_{M+1} = \bigcup_{q \in \mathcal{K}^{M+1}} W_u(q).$$

Each such chain c is spanned by simplices of dimension k. Therefore the support of c is provided by continuous maps σ from the standard k-simplex into I^M. Using a transversality argument (after suitably approximating σ by differentiable maps) we may assume that the image of σ, $\operatorname{Im} \sigma$, is transversal to $W_s(q)$ for all $q \in \mathcal{K}^{M+1}$ where

$$\mathcal{K}^{M+1} = \{q \in \Lambda \mid I'(q) = 0 \quad \text{and} \quad I(q) \leq M+1\}.$$

Therefore

$$\operatorname{Im} \sigma \cap W_s(q) = \phi \tag{9.6}$$

for any $q \in \mathcal{K}^{M+1}$ such that $m(q) > k$. Since the support of c does not meet the stable manifolds of the critical points with $m(q) > k$, $\mathcal{V}_\epsilon(\mathcal{D}_{M+1})$ may be reduced and thus each

chain of $H_k(\Lambda)$ may be represented as a chain in $H_k(\mathcal{W}^\infty \cup \mathcal{V}_\epsilon(A_k \cup B_{k-1}))$. Letting $\epsilon \to 0$ and arguing as for $\mathcal{V}_\epsilon(\mathcal{D}_{M+1})$ in §8, the chains may be represented in $H_k(\mathcal{W}^\infty \cup A_k \cup B_{k-1})$.

The next step in the proof of Theorem 3 is to establish:

$$N_k \geq \beta_k(\Lambda) - \dim H_k(\tilde{\mathcal{W}}^\infty \cup B_{k-1}) \tag{9.7}$$

where $\tilde{\mathcal{W}}^\infty$ will be defined shortly. Set

$$Z_{ij}^\infty = \bigcup_{(\overline{q}_i, \overline{q}_j) \in \mathcal{K}_{ij}} \{(q_i, q_j, Q_r) \in \Lambda_{ij} \times \mathbf{R}^\ell \mid (q_i, q_j) \in W_u(\overline{q}_i, \overline{q}_j) \quad \text{and} \quad \frac{1}{1 + |Q_r - \frac{1}{2}[q_i + q_j]|^2} \leq \frac{\alpha(q_i, q_j)}{4}\}$$

(compare with (5.10)). Using Corollary 3.41 and arguments close to those of §5, it is not difficult to extend $\mathcal{W}^\infty$ to $\tilde{\mathcal{W}}^\infty$, a neighborhood of $\mathcal{C}_1^\infty \equiv I^{\epsilon_1} \cup \left(\bigcup_{i \neq j} Z_{ij}^\infty\right)$ which retracts by deformation on $\mathcal{C}_1^\infty$. This can be done since $\alpha(q_i, q_j) \leq \beta(C_1)$ with α defined in Corollary 3.41. Since $\tilde{\mathcal{W}}^\infty \cap I^{M+1} = \mathcal{W}^\infty$, each chain of $H_k(\Lambda)$ may be represented as a chain of $H_k(\tilde{\mathcal{W}}^\infty \cup A_k \cup B_{k-1})$.

Next we show that

$$H_k(\tilde{\mathcal{W}}^\infty \cup A_k \cup B_{k-1}, \tilde{\mathcal{W}}^\infty \cup B_{k-1}) = \mathbf{Q}^{N_k}. \tag{9.8}$$

To prove (9.8), we employ a variation of an argument of §8 which allowed us to deform I^{M+1} onto $\mathcal{W}^\infty \cup \mathcal{V}_\epsilon$. This argument was based on the transversality of the pseudogradient flow to $\mathcal{W}^\infty$ along its boundary and on the local fibering given by Proposition 7.24 of $W_u(q)$ onto $W_u(q')$ if $W_s(q') \cap W_u(q) \neq \phi$. Observe that $(A_k \backslash \mathcal{K}_k^M) \cup \tilde{\mathcal{W}}^\infty \cup B_{k-1}$ is invariant under the flow (4.23). Moreover by Proposition 7.24, $A_k \cup B_{k-1}$ fibers locally over B_{k-1}. Therefore $(A_k \backslash \mathcal{K}_k^M) \cup \tilde{\mathcal{W}}^\infty \cup B_{k-1}$ has the homotopy type of $\tilde{\mathcal{W}}^\infty \cup B_{k-1}$. Hence

$$\begin{aligned} &H_k(\tilde{\mathcal{W}}^\infty \cup A_k \cup B_{k-1}, \tilde{\mathcal{W}}^\infty \cup B_{k-1}) \\ &\quad = H_k(\tilde{\mathcal{W}}^\infty \cup A_k \cup B_{k-1}, (A_k \backslash \mathcal{K}_k^M) \cup \tilde{\mathcal{W}}^\infty \cup B_{k-1}). \end{aligned} \tag{9.9}$$

By excision,

$$H_k(\tilde{\mathcal{W}}^\infty \cup A_k \cup B_{k-1}, \tilde{\mathcal{W}}^\infty \cup B_{k-1}) = H_k(A_k, A_k \backslash \mathcal{K}_k^M) = \mathbf{Q}^{N_k} \tag{9.10}$$

as claimed above.

Since any generator of $H_k(\Lambda)$ can be represented in $H_k(\tilde{\mathcal{W}}^\infty \cup A_k \cup B_{k-1})$ via the above remarks and this representation is injective,

$$\dim H_k(\tilde{\mathcal{W}}^\infty \cup A_k \cup B_{k-1}) \geq \beta_k(\Lambda). \tag{9.11}$$

Using (9.11) together with the exact sequence for the pairs $(\tilde{\mathcal{W}}^\infty \cup A_k \cup B_{k-1}, \tilde{\mathcal{W}}^\infty \cup B_{k-1})$ yields (9.7).

Next we will prove

$$N_k \geq \beta_k(\Lambda) - \dim H_k(\tilde{\mathcal{W}}^\infty). \tag{9.12}$$

To do so, we first show

$$H_k(B_{k-1}) = 0 \tag{9.13}$$

and

$$H_{k-1}(B'_{k-1} \cap \partial\tilde{\mathcal{W}}^\infty) = 0 \tag{9.14}$$

where $B'_{k-1} = \overline{B_{k-1} \backslash (B_{k-1} \cap \tilde{\mathcal{W}}^\infty)}$. To obtain (9.13)-(9.14), observe that B_{k-1} is the union of manifolds of dimension at most $k-1$ and that by the transversality of the intersection of B_{k-1} with $\mathcal{W}^\infty$ — see the proof of 6° of Theorem 8.2 — $B_{k-1} \cap \partial\tilde{\mathcal{W}}^\infty$ is the union of manifolds of dimension at most $k-2$. We will prove (9.13)-(9.14) by induction on the number of these manifolds. Proposition 7.24 holds for the critical points of B_{k-1} and provides a local fibering of $W_u(x) \cup W_u(y)$ over $W_u(y)$ if $W_u(x) \cap W_s(y) \neq \phi$. It follows that $W_u(y)$ has a neighborhood $\mathcal{V}_y^x$ in $W_u(y) \cup W_u(x)$ which (i) retracts by

deformation on $W_u(y)$ and such that (ii) $W_u(x) \cap \mathcal{V}_y^x$ is open in $W_u(x)$ and distinct from $W_u(x)$. Since $W_u(x)$ is a disk of dimension at most $k-1$ and (ii) holds,

$$H_{k-1}(W_u(x) \cap \mathcal{V}_y^x) = 0. \tag{9.15}$$

Again using the fact that $W_u(x)$ is a disk of dimension at most $k-1$ and (i),

$$H_k(\mathcal{V}_y^x) = 0. \tag{9.16}$$

Therefore using the Mayer-Vietoris sequence applied to the excisive triad $(W_u(x) \cup W_u(y), \mathcal{V}_y^x, W_u(x))$, we see that

$$H_k(W_u(x) \cup W_u(y)) = 0. \tag{9.17}$$

This result extends by induction (based on Proposition 7.24 or more properly a variant of it involving more than two critical points) and leads to (9.13).

Now we turn to the proof of (9.14). The idea behind its proof is the same as the one just employed, but with a shift of one in dimensions. It was pointed out in the proof of 6° of Theorem 8.2 that $\mathcal{W}^\infty$ may be chosen so that the flow (4.23) is transverse to $\mathcal{W}^\infty$ along its boundary. Since this flow is tangent to any $W_u(y)$, the fibrations of Proposition 7.24 are transverse to $\tilde{\mathcal{W}}^\infty$ along its boundary, i.e. if $W_u(y)$ intersects $\partial\tilde{\mathcal{W}}^\infty$, $W_u(y)$ being contained in B_{k-1}, and if $W_u(x) \cap W_s(y) \neq \phi$, $W_u(x)$ also being contained in B_{k-1}, then $(W_u(y) \cup W_u(x)) \cap \partial\tilde{\mathcal{W}}^\infty$ fibers locally over $W_u(y) \cap \partial\tilde{\mathcal{W}}^\infty$ in the sense of Proposition 7.24 with a fiber homeomorphic to $(W_u(x) \cap W_s(y)) \cup \{y\}$.

Each set $W_u(x) \cap \partial\tilde{\mathcal{W}}^\infty$ is a union of manifolds of dimension at most $k-2$ since this intersection is transversal. Using the fibrations as earlier we may construct excisive triads

$$((W_u(x) \cup W_u(y)) \cap \partial\tilde{\mathcal{W}}^\infty,\ \mathcal{V}_y^x \cap \partial\tilde{\mathcal{W}}^\infty,\ W_u(y) \cap \partial\tilde{\mathcal{W}}^\infty)$$

with

$$H_{k-2}(\mathcal{V}_y^x \cap W_u(x) \cap \partial\tilde{\mathcal{W}}^\infty) = 0 \tag{9.18}$$

and

$$H_{k-1}(\mathcal{V}_y^x \cap \partial\tilde{\mathcal{W}}^\infty) = 0. \tag{9.19}$$

Therefore arguing as for (9.13),

$$H_{k-1}(B_{k-1} \cap \partial\tilde{\mathcal{W}}^\infty) = 0. \tag{9.20}$$

Using the fact that the flow (4.23) is transverse to $\partial\mathcal{W}^\infty$, B_{k-1} may be retracted by deformation onto B'_{k-1}. Hence $(\mathcal{W}^\infty \cup B_{k-1}, \tilde{\mathcal{W}}^\infty, B'_{k-1})$ is excisive,

$$H_k(B'_{k-1}) = 0, \tag{9.21}$$

and (9.14) holds. Therefore the Mayer-Vietoris sequence implies

$$\dim H_k(\tilde{\mathcal{W}}^\infty \cup B_{k-1}) \leq \dim H_k(\tilde{\mathcal{W}}^\infty). \tag{9.22}$$

Combining (9.22) and (9.7) yields (9.12).

For the final step of the proof of Theorem 3, note that by the arguments of §8 for $\mathcal{W}^\infty$, $\tilde{\mathcal{W}}^\infty$ has the same homotopy type as $\mathcal{C}_1^\infty$. Then by similar arguments to those used for $\mathcal{C}_1$ in §5, for $k \geq 3\ell + 1$,

$$H_k(\tilde{\mathcal{W}}^\infty) = H_k(\mathcal{C}_1^\infty) = \bigoplus_{i<j} H_k(B_{ij}^\infty) \tag{9.23}$$

where $B_{ij}^\infty = (Z_{ij}^\infty \backslash \text{int } I^{\epsilon_1}) \cup W_{ij}^{\epsilon_1}$ and $W_{ij}^{\epsilon_1}$ was defined following (5.6). The set B_{ij}^∞ has the homotopy type of $S^{\ell-1} \times \mathcal{L}_{ij}$ where

$$\begin{aligned} \mathcal{L}_{ij} = \cup_{(\overline{q}_i, \overline{q}_j) \in \mathcal{K}_{ij}} &\{(q_i, q_j) \in W_u(\overline{q}_i, \overline{q}_j) \mid \epsilon_1 - \frac{\alpha(q_1, q_2)}{4} \leq J_{ij}(q_i, q_j)\} \\ &\cup \{(q_i, q_j) \in \Lambda_{ij} \mid \epsilon_1 - \frac{\alpha(q_i, q_j)}{4} \leq J_{ij}(q_i, q_j) < \epsilon_1\}. \end{aligned} \tag{9.24}$$

Using (5.6)-(5.8), $\mathcal{L}_{ij}$ is a retract by deformation of

$$\mathcal{L}'_{ij} = \left(\bigcup_{(\overline{q}_i, \overline{q}_j) \in \mathcal{K}_{ij}} W_u(\overline{q}_i, \overline{q}_j) \right) \cup \text{ int } J_{ij}^{\epsilon_1}. \tag{9.25}$$

Therefore using the pseudogradient flow (4.23) again, $\mathcal{L}_{ij}$ has the homotopy type of

$$\mathcal{L}''_{ij} = J_{ij}^{\epsilon_1/2} \cup \left(\bigcup_{(\bar{q}_i,\bar{q}_j)\in\mathcal{K}_{ij}} W_u(\bar{q}_i,\bar{q}_j) \right). \tag{9.26}$$

Applying Theorem 7.2 (or actually a generalization to the analogous infinite dimensional compact and Fredholm framework — see e.g. (15)) we obtain that $\mathcal{L}''_{ij}$ has the homotopy type of Λ_{ij}. Therefore B_{ij}^{∞} has the homotopy type of $\Lambda_{ij} \times S^{\ell-1}$ and

$$\dim H_k(B_{ij}^{\infty}) = \dim H_{k+\ell-1}(\Lambda_{ij}) + \dim H_k(\Lambda_{ij}). \tag{9.27}$$

As was shown in §5, Λ_{ij} has the homotopy type of the loop space of $S^{\ell-1}$. Therefore

$$\dim H_k(B_{ij}^{\infty}) \leq 4 \tag{9.28}$$

and by (9.23)

$$\dim H_k(\tilde{W}^{\infty}) \leq 12 \tag{9.29}$$

for $k \geq 3\ell + 1$. Combining (9.12) and (9.29) yields Theorem 3.

Remark 9.30. As mentioned in the Introduction, Theorem 3 has consequences for e.g. homogeneous potential like those yielding central configuration solutions (under (V_6)). Modulo scaling, these special solutions are *generated by a compact family* of solutions. Therefore the contribution of the whole family (after scaling) to the homology groups of Λ is bounded. Hence there must exist periodic solutions other than these special solutions. The same argument may be applied to treat the multiplicity of hyperbolic or elliptic solutions of fixed energy as mentioned in the Introduction. These results will be pursued elsewhere.

Remark 9.31. If V is autonomous, the requirement in Theorem 3 that $\overline{m}(q) - m(q) = \ell$ cannot be satisfied. Indeed the resulting S^1 invariance of I implies that critical points occur in circles. Thus $\overline{m}(q) - m(q) \geq \ell + 1$ for any critical point q of I. In this setting we can de.'.e $\overline{N}_k$, the number of critical circles of Morse index k. Now we have:

Corollary 9.32. Assume V satisfies $(V_1)-(V_6)$ and V is autonomous. Suppose that any critical point q of I with $m(q) \leq k$ or $\overline{m}(q) \geq k$ satisfies $\overline{m}(q) - m(q) = \ell + 1$. Then

$$\overline{N}_k + \overline{N}_{k-1} \geq \beta_k(\Lambda) - 12 \tag{9.33}$$

for $k \geq 3\ell + 1$.

Proof. By perturbing I slightly, any circle of critical points of I can be broken up into two critical points, one of Morse index k and the other of Morse index $k+1$. Doing this for each circle of critical points of I and applying the argument of Theorem 3 yields (9.33).

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